Section A: Pure Mathematics

1 Find the integer, n, that satisfies $n^2 < 33\,127 < (n+1)^2$. Find also a small integer m such that $(n+m)^2 - 33\,127$ is a perfect square. Hence express $33\,127$ in the form pq, where p and q are integers greater than 1.

By considering the possible factorisations of $33\,127$, show that there are exactly two values of m for which $(n+m)^2 - 33\,127$ is a perfect square, and find the other value.

2 A small goat is tethered by a rope to a point at ground level on a side of a square barn which stands in a large horizontal field of grass. The sides of the barn are of length $2a$ and the rope is of length $4a$. Let A be the area of the grass that the goat can graze. Prove that $A \leqslant 14\pi a^2$ and determine the minimum value of A.

3 In this question b, c, p and q are real numbers.

(i) By considering the graph $y = x^2 + bx + c$ show that $c < 0$ is a sufficient condition for the equation $x^2 + bx + c = 0$ to have distinct real roots. Determine whether $c < 0$ is a necessary condition for the equation to have distinct real roots.

(ii) Determine necessary and sufficient conditions for the equation $x^2 + bx + c = 0$ to have distinct positive real roots.

(iii) What can be deduced about the number and the nature of the roots of the equation $x^3 + px + q = 0$ if $p > 0$ and $q < 0$?

What can be deduced if $p < 0$ and $q < 0$? You should consider the different cases that arise according to the value of $4p^3 + 27q^2$.

4 By sketching on the same axes the graphs of $y = \sin x$ and $y = x$, show that, for $x > 0$:

(i) $x > \sin x$;

(ii) $\dfrac{\sin x}{x} \approx 1$ for small x.

A regular polygon has n sides, and perimeter P. Show that the area of the polygon is

$$\frac{P^2}{4n\tan\left(\dfrac{\pi}{n}\right)} .$$

Show by differentiation (treating n as a continuous variable) that the area of the polygon increases as n increases with P fixed.

Show also that, for large n, the ratio of the area of the polygon to the area of the smallest circle which can be drawn around the polygon is approximately 1.

5 **(i)** Use the substitution $u^2 = 2x + 1$ to show that, for $x > 4$,

$$\int \frac{3}{(x-4)\sqrt{2x+1}}\,\mathrm{d}x = \ln\left(\frac{\sqrt{2x+1}-3}{\sqrt{2x+1}+3}\right) + K ,$$

where K is a constant.

(ii) Show that $\displaystyle\int_{\ln 3}^{\ln 8} \frac{2}{\mathrm{e}^x\sqrt{\mathrm{e}^x+1}}\,\mathrm{d}x = \frac{7}{12} + \ln\frac{2}{3}$.

6 **(i)** Show that, if $(a\,,b)$ is **any** point on the curve $x^2 - 2y^2 = 1$, then $(3a + 4b\,, 2a + 3b)$ also lies on the curve.

(ii) Determine the smallest positive integers M and N such that, if $(a\,,b)$ is **any** point on the curve $Mx^2 - Ny^2 = 1$, then $(5a + 6b\,, 4a + 5b)$ also lies on the curve.

(iii) Given that the point $(a\,,b)$ lies on the curve $x^2 - 3y^2 = 1$, find positive integers P, Q, R and S such that the point $(Pa + Qb\,, Ra + Sb)$ also lies on the curve.

7 **(i)** Sketch on the same axes the functions $\operatorname{cosec} x$ and $2x/\pi$, for $0 < x < \pi$. Deduce that the equation $x \sin x = \pi/2$ has exactly two roots in the interval $0 < x < \pi$.

Show that

$$\int_{\pi/2}^{\pi} \left| x \sin x - \frac{\pi}{2} \right| \, \mathrm{d}x = 2 \sin \alpha + \frac{3\pi^2}{4} - \alpha\pi - \pi - 2\alpha \cos \alpha - 1$$

where α is the larger of the roots referred to above.

(ii) Show that the region bounded by the positive x-axis, the y-axis and the curve

$$y = \Big| |\mathrm{e}^x - 1| - 1 \Big|$$

has area $\ln 4 - 1$.

8 *Note that the volume of a tetrahedron is equal to* $\frac{1}{3} \times$ *the area of the base* $\times$ *the height.*

The points O, A, B and C have coordinates $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, respectively, where a, b and c are positive.

(i) Find, in terms of a, b and c, the volume of the tetrahedron $OABC$.

(ii) Let angle $ACB = \theta$. Show that

$$\cos \theta = \frac{c^2}{\sqrt{(a^2 + c^2)(b^2 + c^2)}}$$

and find, in terms of a, b and c, the area of triangle ABC.

Hence show that d, the perpendicular distance of the origin from the triangle ABC, satisfies

$$\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \, .$$

Section B: Mechanics

9 A block of mass $4\,$kg is at rest on a smooth, horizontal table. A smooth pulley P is fixed to one edge of the table and a smooth pulley Q is fixed to the opposite edge. The two pulleys and the block lie in a straight line.

Two horizontal strings are attached to the block. One string runs over pulley P; a particle of mass $x\,$kg hangs at the end of this string. The other string runs over pulley Q; a particle of mass $y\,$kg hangs at the end of this string, where $x > y$ and $x + y = 6$.

The system is released from rest with the strings taut. When the $4\,$kg block has moved a distance d, the string connecting it to the particle of mass $x\,$kg is cut. Show that the time taken by the block from the start of the motion until it first returns to rest (assuming that it does not reach the edge of the table) is $\sqrt{d/(5g)}\;\mathrm{f}(y)$, where

$$\mathrm{f}(y) = \frac{10}{\sqrt{6-2y}} + \left(1 + \frac{4}{y}\right)\sqrt{6-2y}.$$

Calculate the value of y for which $\mathrm{f}'(y) = 0$.

10 A particle P is projected in the x-y plane, where the y-axis is vertical and the x-axis is horizontal. The particle is projected with speed V from the origin at an angle of 45° above the positive x-axis. Determine the equation of the trajectory of P.

The point of projection (the origin) is on the floor of a barn. The roof of the barn is given by the equation $y = x\tan\alpha + b$, where $b > 0$ and α is an acute angle. Show that, if the particle just touches the roof, then $V(-1+\tan\alpha) = -2\sqrt{bg}$; you should justify the choice of the negative root. If this condition is satisfied, find, in terms of α, V and g, the time after projection at which touching takes place.

A particle Q can slide along a smooth rail fixed, in the x-y plane, to the under-side of the roof. It is projected from the point $(0, b)$ with speed U at the same time as P is projected from the origin. Given that the particles just touch in the course of their motions, show that

$$2\sqrt{2}\,U\cos\alpha = V\big(2 + \sin\alpha\cos\alpha - \sin^2\alpha\big).$$

11 Particles A_1, A_2, A_3, ..., A_n (where $n \geqslant 2$) lie at rest in that order in a smooth straight horizontal trough. The mass of A_{n-1} is m and the mass of A_n is λm, where $\lambda > 1$. Another particle, A_0, of mass m, slides along the trough with speed u towards the particles and collides with A_1. Momentum and energy are conserved in all collisions.

(i) Show that it is not possible for there to be exactly one particle moving after all collisions have taken place.

(ii) Show that it is not possible for A_{n-1} and A_n to be the only particles moving after all collisions have taken place.

(iii) Show that it is not possible for A_{n-2}, A_{n-1} and A_n to be the only particles moving after all collisions have taken place.

(iv) Given that there are exactly two particles moving after all collisions have taken place, find the speeds of these particles in terms of u and λ.

Section C: Probability and Statistics

12 Oxtown and Camville are connected by three roads, which are at risk of being blocked by flooding. On two of the three roads there are two sections which may be blocked. On the third road there is only one section which may be blocked. The probability that each section is blocked is p. Each section is blocked independently of the other four sections. Show that the probability that Oxtown is cut off from Camville is $p^3\,(2-p)^2$.

I want to travel from Oxtown to Camville. I choose one of the three roads at random and find that my road is not blocked. Find the probability that I would not have reached Camville if I had chosen either of the other two roads. You should factorise your answer as fully as possible.

Comment briefly on the value of this probability in the limit $p \to 1$.

13 A very generous shop-owner is hiding small diamonds in chocolate bars. Each diamond is hidden independently of any other diamond, and on average there is one diamond per kilogram of chocolate.

(i) I go to the shop and roll a fair six-sided die once. I decide that if I roll a score of N, I will buy $100N$ grams of chocolate. Show that the probability that I will have no diamonds is

$$\frac{\mathrm{e}^{-0.1}}{6}\left(\frac{1-\mathrm{e}^{-0.6}}{1-\mathrm{e}^{-0.1}}\right)$$

Show also that the expected number of diamonds I find is 0.35.

(ii) Instead, I decide to roll a fair six-sided die repeatedly until I score a 6. If I roll my first 6 on my Tth throw, I will buy $100T$ grams of chocolate. Show that the probability that I will have no diamonds is

$$\frac{\mathrm{e}^{-0.1}}{6-5\mathrm{e}^{-0.1}}$$

Calculate also the expected number of diamonds that I find. (You may find it useful to consider the the binomial expansion of $(1-x)^{-2}$.)

14 **(i)** A bag of sweets contains one red sweet and n blue sweets. I take a sweet from the bag, note its colour, return it to the bag, then shake the bag. I repeat this until the sweet I take is the red one. Find an expression for the probability that I take the red sweet on the rth attempt. What value of n maximises this probability?

(ii) Instead, I take sweets from the bag, without replacing them in the bag, until I take the red sweet. Find an expression for the probability that I take the red sweet on the rth attempt. What value of n maximises this probability?