STEP Support Programme

2025 STEP 3 Worked Paper

General comments

These solutions have a lot more words in them than you would expect to see in an exam script and in places I have tried to explain some of my thought processes as I was attempting the questions. What you will not find in these solutions is my crossed out mistakes and wrong turns, but please be assured that they did happen!

You can find the examiners report and mark schemes for this paper from the OCR website. These are the general comments for the STEP 3 2025 exam from the Examiner's report:

The majority of candidates focused solely on the pure questions, with questions 1, 2 and 8 the most popular. The statistics questions were more popular than the mechanics questions in this exam series.

Candidates who did well on this paper generally:

- *were careful to explain and justify the steps in their arguments, explaining what they had done rather than expecting the examiner to infer what had been done from disjointed groups of calculations*
- *paid close attention to what was required by the questions*
- *made fewer unnecessary mistakes with calculations*
- *thought carefully about how to present rigorous arguments involving trig functions and their inverse functions, especially in relation to domain considerations*
- *understood that questions set on the STEP papers require sufficient justification to earn full credit*
- *knew the difference between 'positive' and 'non-negative'*
- *attempted all parts of a question, picking up marks for later parts even when they had not necessarily attempted or completed previous parts.*

Candidates who did less well on this paper generally:

- *did not pay attention to 'Hence' instructions: this means that you must use the previous part*
- *presented explanations that were not precise enough (e.g. in Question 3 describing the transformations but not in the context of the graphs or in Question 8 not explaining use of trigonometric relationships sufficiently well)*
- *made additional assumptions, e.g. that a function was differentiable when this had not been given*

- *tried to present if and only if arguments in a single argument when dealing with each direction separately would have been more appropriate and safer (note that this is not always the case; in general candidates need to consider what is the most appropriate presentation of an if and only if argument)*
- *tried to carry out too many steps in one go, resulting in them not justifying the key steps sufficiently*
- *did not take sufficient care with graphs/curve sketching.*

Please send any corrections, comments or suggestions to step@maths.cam.ac.uk.

Question 1

1 *You need not consider the convergence of the improper integrals in this question.*

For $p, q > 0$, define

$$\mathrm{b}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x.$$

(i) Show that $\mathrm{b}(p,q) = \mathrm{b}(q,p)$.

(ii) Show that $\mathrm{b}(p+1,q) = \mathrm{b}(p,q) - \mathrm{b}(p,q+1)$ and hence that $\mathrm{b}(p+1,p) = \frac{1}{2}\,\mathrm{b}(p,p)$.

(iii) Show that

$$\mathrm{b}(p,q) = 2\int_0^{\frac{1}{2}\pi} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\,\mathrm{d}\theta.$$

Hence show that $\mathrm{b}(p,p) = \dfrac{1}{2^{2p-1}}\,\mathrm{b}\big(p,\frac{1}{2}\big)$.

(iv) Show that

$$\mathrm{b}(p,q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\,\mathrm{d}t.$$

(v) Evaluate

$$\int_0^\infty \frac{t^{\frac{3}{2}}}{(1+t)^6}\,\mathrm{d}t.$$

Examiner's report

This question was the most popular question in terms of the number of attempts, and it was generally well done. Some candidates spent significant time attempting methods involving integration by parts in the early parts of this question which did not work. In part **(iii)** the most common method in the 'Hence show that...' involved using the substitution $u = 2\theta$ at some point. After this the integral looks like what is required in the given answer but with the limits 0 to π rather than 0 to $\frac{1}{2}\pi$. Candidates needed to point out that the symmetry in the integral enabled the limits to be changed back to 0 to $\frac{1}{2}\pi$ with the appearance of a factor of 2.

Part **(iv)** was generally well done by those who got that far. In part **(v)** some marks were given for piecing together the earlier results to get to an integral that is relatively easy to calculate. Some candidates did not provide sufficient justification to be awarded full credit. A good number of candidates got to a final correct value of the integral. Some candidates had success with alternative methods involving more difficult integration having not made so much use of the properties of b to simplify the calculation.

Solution

(i) Using a substitution of $t = 1 - x$ we have:

$$\begin{aligned}
\mathrm{b}(p,q) &= \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x \\
&= \int_1^0 (1-t)^{p-1}t^{q-1}\,(-1)\times\mathrm{d}t \\
&= \int_0^1 (1-t)^{p-1}t^{q-1}\,\mathrm{d}t \\
&= \mathrm{b}(q,p)
\end{aligned}$$

(ii) Starting with the right hand side we have:

$$\begin{aligned}
\mathrm{b}(p,q) - \mathrm{b}(p,q+1) &= \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x - \int_0^1 x^{p-1}(1-x)^{q}\,\mathrm{d}x \\
&= \int_0^1 \Big[x^{p-1}(1-x)^{q-1} - x^{p-1}(1-x)^{q}\Big]\,\mathrm{d}x \\
&= \int_0^1 x^{p-1}(1-x)^{q-1}\Big[1-(1-x)\Big]\,\mathrm{d}x \\
&= \int_0^1 x^{p-1}(1-x)^{q-1}\times x\,\mathrm{d}x \\
&= \int_0^1 x^{p}(1-x)^{q-1}\,\mathrm{d}x \\
&= \mathrm{b}(p+1,q)
\end{aligned}$$

We have:

$$\begin{aligned}
\mathrm{b}(p+1,q) &= \mathrm{b}(p,q) - \mathrm{b}(p,q+1) \\
&= \mathrm{b}(p,q) - \mathrm{b}(q+1,p) \quad \text{using } \textbf{(i)} \\
\implies \mathrm{b}(p+1,p) &= \mathrm{b}(p,p) - \mathrm{b}(p+1,p) \\
\implies \mathrm{b}(p+1,p) &= \frac{1}{2}\mathrm{b}(p,p)
\end{aligned}$$

(iii) In this part we are aiming to get to $\mathrm{b}(p,p) = \dfrac{1}{2^{2p-1}}\mathrm{b}(p,\tfrac{1}{2}) = \dfrac{1}{2^{2p-1}}\displaystyle\int_0^1 x^{p-1}(1-x)^{\frac{1}{2}-1}\mathrm{d}x$. It is often a good idea to write down what you are aiming for at the start of the question!

Using a substitution of $x = \sin^2\theta$ in the integral given in the stem of the question gives:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}\theta} &= 2\sin\theta\cos\theta \\
\mathrm{b}(p,q) &= \int_0^{\frac{1}{2}\pi} \big(\sin^2\theta\big)^{p-1}\big(1-\sin^2\theta\big)^{q-1} 2\sin\theta\cos\theta\,\mathrm{d}\theta \\
&= 2\int_0^{\frac{1}{2}\pi} \big(\sin\theta\big)^{2p-1}\big(\cos\theta\big)^{2q-1}\,\mathrm{d}\theta
\end{aligned}$$

Setting $p = q$ gives:

$$\begin{aligned}
\mathrm{b}(p,p) &= 2\int_0^{\frac{1}{2}\pi} \big(\sin\theta\big)^{2p-1}\big(\cos\theta\big)^{2p-1}\,\mathrm{d}\theta \\
&= 2\int_0^{\frac{1}{2}\pi} \big(\sin\theta\cos\theta\big)^{2p-1}\,\mathrm{d}\theta \\
&= 2\int_0^{\frac{1}{2}\pi} \left(\frac{\sin 2\theta}{2}\right)^{2p-1}\,\mathrm{d}\theta \\
&= \frac{2}{2^{2p-1}}\int_0^{\frac{1}{2}\pi} (\sin 2\theta)^{2p-1}\,\mathrm{d}\theta
\end{aligned}$$

Using a substitution of $t = 2\theta$ gives:

$$\begin{aligned}
\mathrm{b}(p,p) &= \frac{2}{2^{2p-1}}\int_0^{\frac{1}{2}\pi} (\sin 2\theta)^{2p-1}\,\mathrm{d}\theta \\
&= \frac{2}{2^{2p-1}}\int_0^{\pi} (\sin t)^{2p-1}\,\tfrac{1}{2}\mathrm{d}t \\
&= \frac{1}{2^{2p-1}}\int_0^{\pi} (\sin t)^{2p-1}\,\mathrm{d}t
\end{aligned}$$

Since $\sin x$ is symmetrical about $x = \frac{1}{2}\pi$ we can rewrite this as:

$$\mathrm{b}(p,p) = \frac{2}{2^{2p-1}}\int_0^{\frac{1}{2}\pi} (\sin t)^{2p-1}\,\mathrm{d}t$$

Then using $x = \sin^2 t$ we have:

$$\begin{aligned}
\mathrm{b}(p,p) &= \frac{2}{2^{2p-1}}\int_0^{\frac{1}{2}\pi} (\sin t)^{2p-1}\,\mathrm{d}t \\
&= \frac{2}{2^{2p-1}}\int_0^{1} (\sin t)^{2p-1}\,\frac{1}{2\sin t\cos t}\mathrm{d}x \\
&= \frac{1}{2^{2p-1}}\int_0^{1} (\sin t)^{2p-2}\,\cos t^{-1}\mathrm{d}x \\
&= \frac{1}{2^{2p-1}}\int_0^{1} \big(\sin^2 t\big)^{p-1}\,(1-\sin^2 t)^{-\frac{1}{2}}\mathrm{d}x \\
&= \frac{1}{2^{2p-1}}\int_0^{1} x^{p-1}\,(1-x)^{-\frac{1}{2}}\mathrm{d}x \\
&= \frac{1}{2^{2p-1}}\mathrm{b}(p,\tfrac{1}{2})
\end{aligned}$$

as required

(iv) Let $x = \dfrac{1}{u}$:

$$\begin{aligned} \mathrm{b}(p,q) &= \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x \\ &= \int_\infty^1 \left(\frac{1}{u}\right)^{p-1}\left(1-\frac{1}{u}\right)^{q-1} \times \frac{-1}{u^2}\,\mathrm{d}u \\ &= \int_1^\infty \left(\frac{1}{u}\right)^{p-1}\left(\frac{u-1}{u}\right)^{q-1} \times \frac{1}{u^2}\,\mathrm{d}u \\ &= \int_1^\infty \frac{(u-1)^{q-1}}{u^{p+q-2}\times u^2}\,\mathrm{d}u \\ &= \int_1^\infty \frac{(u-1)^{q-1}}{u^{p+q}}\,\mathrm{d}u \end{aligned}$$

Then using $u-1=t$ we have:

$$\begin{aligned} \mathrm{b}(p,q) &= \int_1^\infty \frac{(u-1)^{q-1}}{u^{p+q}}\,\mathrm{d}u \\ &= \int_0^\infty \frac{t^{q-1}}{(1+t)^{p+q}}\,\mathrm{d}t \end{aligned}$$

and by using part **(i)** we have:

$$\begin{aligned} \mathrm{b}(p,q) &= \mathrm{b}(q,p) \\ &= \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\,\mathrm{d}t \qquad \text{by swapping } p,q \end{aligned}$$

(v) Comparing $\displaystyle\int_0^\infty \frac{t^{\frac{3}{2}}}{(1+t)^6}\,\mathrm{d}t$ to $\displaystyle\int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\,\mathrm{d}t$ we have $p=\frac{5}{2}$ and $q=\frac{7}{2}$.

So we want:

$$\begin{aligned} \mathrm{b}\left(\tfrac{5}{2},\tfrac{7}{2}\right) &= \mathrm{b}\left(\tfrac{7}{2},\tfrac{5}{2}\right) && \text{using} \quad \mathrm{b}(p,q)=\mathrm{b}(q,p) \\ &= \tfrac{1}{2}\mathrm{b}\left(\tfrac{5}{2},\tfrac{5}{2}\right) && \text{using} \quad \mathrm{b}(p+1,p)=\tfrac{1}{2}\mathrm{b}(p,p) \\ &= \tfrac{1}{2}\times\tfrac{1}{2^4}\mathrm{b}\left(\tfrac{5}{2},\tfrac{1}{2}\right) && \text{using} \quad \mathrm{b}(p,p)=\tfrac{1}{2^{2p-1}}\mathrm{b}\left(p,\tfrac{1}{2}\right) \end{aligned}$$

We could have started from $\mathrm{b}(p,q) = \displaystyle\int_0^\infty \frac{t^{q-1}}{(1+t)^{p+q}}\,\mathrm{d}t$, but it felt slightly safer to start with the last result given at the end of part **(iv)**.

So the integral we want to evaluate is:

$$
\begin{aligned}
&\frac{1}{2^5}\int_0^1 x^{\frac{5}{2}-1}(1-x)^{\frac{1}{2}-1}\,\mathrm{d}x \\
=&\frac{1}{2^5}\int_0^1 x^{\frac{3}{2}}(1-x)^{-\frac{1}{2}}\,\mathrm{d}x \\
=&\frac{1}{2^5}\int_0^{\frac{1}{2}\pi} \sin^3\theta(1-\sin^2\theta)^{-\frac{1}{2}} \times 2\sin\theta\cos\theta\,\mathrm{d}\theta \qquad \text{using } x=\sin^2\theta \\
=&\frac{1}{2^4}\int_0^{\frac{1}{2}\pi} \sin^4\theta\,\mathrm{d}\theta \\
=&\frac{1}{2^6}\int_0^{\frac{1}{2}\pi} (1-\cos 2\theta)^2\,\mathrm{d}\theta \qquad \text{using } \cos 2A = 1-2\sin^2 A \\
=&\frac{1}{2^6}\int_0^{\frac{1}{2}\pi} 1-2\cos 2\theta+\cos^2 2\theta\,\mathrm{d}\theta \\
=&\frac{1}{2^6}\int_0^{\frac{1}{2}\pi} 1-2\cos 2\theta+\tfrac{1}{2}\left(1+\cos 4\theta\right)\,\mathrm{d}\theta \qquad \text{using } \cos 2A = 2\cos^2 A-1 \\
=&\frac{1}{2^6}\left[\tfrac{3}{2}\theta-\sin 2\theta+\tfrac{1}{8}\sin 4\theta\right]_0^{\frac{1}{2}\pi} \\
=&\frac{1}{2^6}\times\frac{3}{2}\times\frac{\pi}{2} \qquad \text{note that } \sin\pi=\sin 2\pi=\sin 0=0 \\
=&\frac{3\pi}{256}
\end{aligned}
$$

Question 2

2 Let $\mathrm{f}(x) = 7 - 2|x|$.

A sequence $u_0, u_1, u_2, \ldots$ is defined by $u_0 = a$ and $u_n = \mathrm{f}(u_{n-1})$ for $n > 0$.

(i) **(a)** Sketch, on the same axes, the graphs with equations $y = \mathrm{f}(x)$ and $y = \mathrm{f}(\mathrm{f}(x))$.

(b) Find all solutions of the equation $\mathrm{f}(\mathrm{f}(x)) = x$.

(c) Find the values of a for which the sequence $u_0, u_1, u_2, \ldots$ has period 2.

(d) Show that, if $a = \frac{28}{5}$, then the sequence $u_2, u_3, u_4, \ldots$ has period 2, but neither u_0 or u_1 is equal to either of u_2 or u_3.

(ii) **(a)** Sketch, on the same axes, the graphs with equations $y = \mathrm{f}(x)$ and $y = \mathrm{f}(\mathrm{f}(\mathrm{f}(x)))$.

(b) Consider the sequence $u_0, u_1, u_2, \ldots$ in the cases $a = 1$ and $a = -\frac{7}{9}$. Hence find all the solutions of the equation $\mathrm{f}(\mathrm{f}(\mathrm{f}(x))) = x$.

(c) Find a value of a such that the sequence $u_3, u_4, u_5, \ldots$ has period 3, but where none of u_0, u_1 or u_2 is equal to any of u_3, u_4 or u_5.

Examiner's report

In terms of attempts, this was the third most popular question. Most candidates who attempted this question were able to make good progress with many of the parts. Candidates were generally able to sketch the graph of $y = \mathrm{f}(x)$, but sketches of $y = \mathrm{f}\big(\mathrm{f}(x)\big)$ often had some features missing or incorrect. Many candidates opted to work out the equation for each of the straight-line segments before sketching the graph and, while this generally resulted in the correct overall shape, important points such as the vertical positioning of the points where the two graphs cross were often incorrect. A number of candidates would have benefited from making their sketches larger. A small number of candidates did not sketch the two graphs on the same set of axes, which meant that some of the marks for this part of the question were not accessible.

Part **(c)** was not answered well, with many candidates simply restating their solutions to part **(b)** without considering the fact that some of the solutions would lead to sequences with a period of 1. Allowance was made here for those candidates who stated that a solution with period 1 also has period 2 and listed all their solutions to **(b)**.

In part **(d)** candidates successfully calculated the terms of the sequence and many identified the connection with the previous parts to explain that the remainder of the sequence would have a period of 2. A small number of candidates only checked that $u_0 \neq u_2$ and $u_1 \neq u_3$ and therefore did not fully answer this part of the question.

Those who had made good sketches for the graphs in part **(i)(a)** generally made good attempts at the sketches in part **(ii)(a)**, although similar issues were encountered with the positioning of the intersections of the two graphs.

Almost all candidates who attempted part **(ii)(b)** were able to calculate the sequences starting with the two given values, although many did not realise that all three values that appeared in each solution would also be solutions. The question was posed using 'Hence' and so this approach was required. Some candidates simply stated their set of solutions without providing any explanation. While some candidates commented on the fact that there must be eight solutions based on their sketch, a significant number of candidates did not realise that the two period 1 values that lead to a constant sequence would also be solutions.

Solution

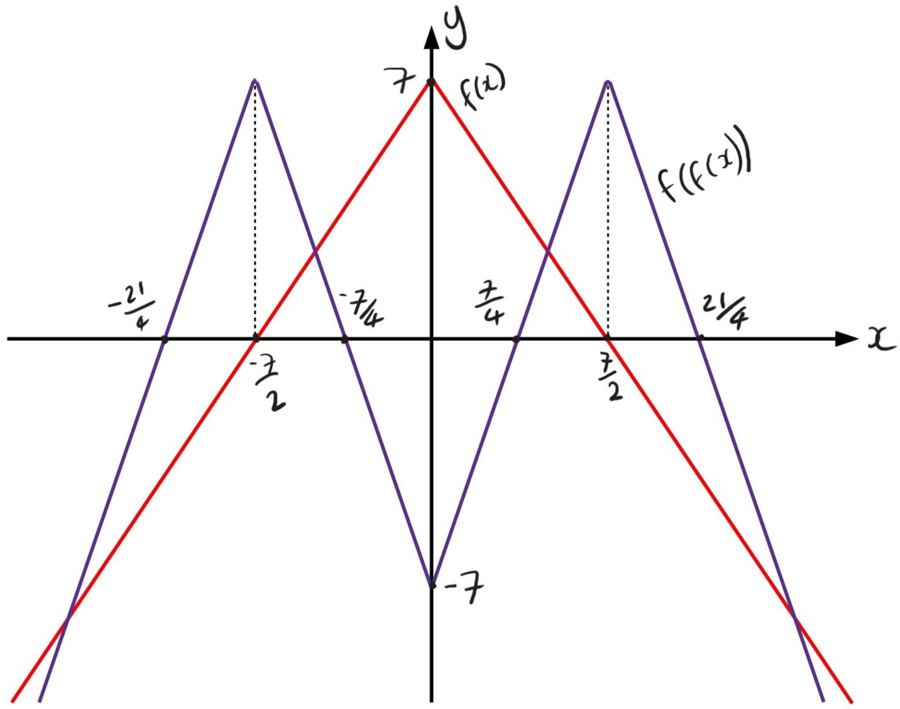
(i) **(a)** The first graph can be drawn by using a series of transformations of $y = |x|$. You also know that it will cross the y axis at $(0, 7)$ and will cross the x axis at $\left(\pm\frac{7}{2}, 0\right)$.

For $y = \mathrm{f}(\mathrm{f}(x))$ you can use the graph of $y = \mathrm{f}(x)$ to help. You know that $\mathrm{f}(\mathrm{f}(0)) = \mathrm{f}(7) = -7$ and $\mathrm{f}\left(\mathrm{f}\left(\pm\frac{7}{2}\right)\right) = \mathrm{f}(0) = 7$.

Where the graph of $\mathrm{f}(\mathrm{f}(x))$ crosses the x axis we have $\mathrm{f}(x) = \pm\frac{7}{2}$. This gives:

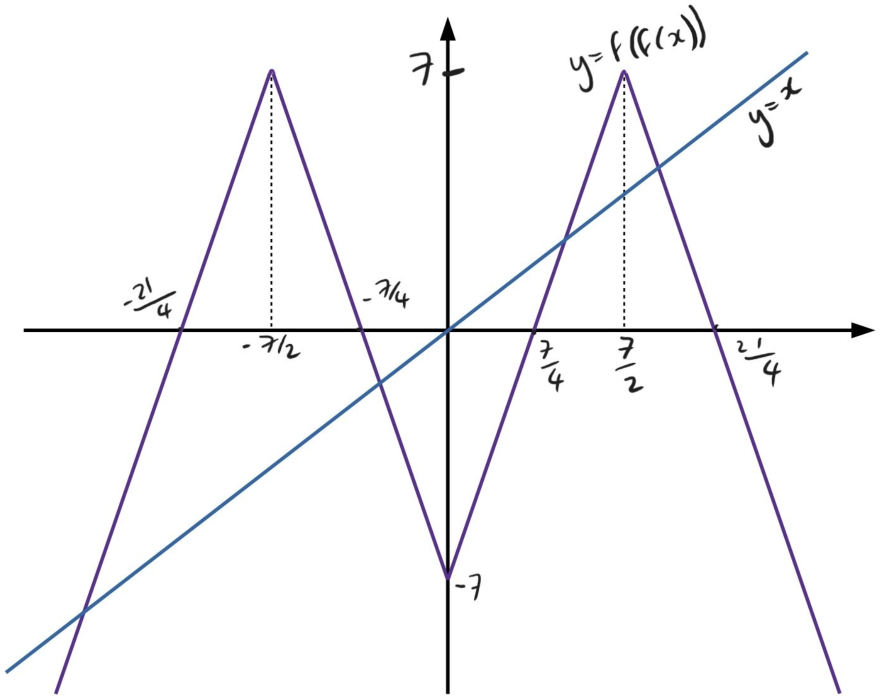
$$\begin{aligned} 7 - 2|x| &= \tfrac{7}{2} \\ 2|x| &= 7 - \tfrac{7}{2} \\ 2|x| &= \tfrac{7}{2} \\ |x| &= \tfrac{7}{4} \\ x &= \pm\tfrac{7}{4} \\ \text{and} \quad 7 - 2|x| &= -\tfrac{7}{2} \\ 2|x| &= 7 + \tfrac{7}{2} \\ 2|x| &= \tfrac{21}{2} \\ x &= \pm\tfrac{21}{4} \end{aligned}$$

The graphs look like:



A ruler is highly recommended for this sketch!

(b) A consideration of the sketch shows that we expect 4 solutions. This is a good technique for checking how many solutions you expect, and seeing if your solutions take reasonable values.



The equations of the "sections" of $\mathrm{f}(\mathrm{f}(x))$ are:

$$\begin{aligned} y &= 4x + 21 && \text{for} && x \leqslant -\tfrac{7}{2} \\ y &= -4x - 7 && \text{for} && -\tfrac{7}{2} \leqslant x \leqslant 0 \\ y &= 4x - 7 && \text{for} && 0 \leqslant x \leqslant \tfrac{7}{2} \\ y &= 21 - 4x && \text{for} && x \geqslant \tfrac{7}{2} \end{aligned}$$

These equations can be written down pretty much straight from the graph if your graph is clear enough! Otherwise you can use the maximum/minimum and intercept points on the graph to find the equations.

Solving for x gives:

$$\begin{aligned} x = 4x + 21 &\implies x = -7 \\ x = -4x - 7 &\implies x = -\tfrac{7}{5} \\ x = 4x - 7 &\implies x = \tfrac{7}{3} \\ x = 21 - 4x &\implies x = \tfrac{21}{5} \end{aligned}$$

Note that all of these solutions fall into the "correct" range for the line segments, and look reasonable when compared to the graph on the previous page.

Alternatively you can ignore the graph and use some algebra (though I did get the question wrong first time I used this method!)

Setting $\mathrm{f}(\mathrm{f}(x)) = x$ gives:

$$\begin{aligned} 7 - 2\big|7 - 2|x|\big| &= x \\ 7 - x &= 2\big|7 - 2|x|\big| \\ 49 - 14x + x^2 &= 4(49 - 28|x| + 4x^2) \quad \text{squaring} \end{aligned}$$

If we take $x \geqslant 0$ this becomes:

$$\begin{aligned} 49 - 14x + x^2 &= 4(49 - 28x + 4x^2) \\ 15x^2 - 98x + 147 &= 0 \\ (5x - 21)(3x - 7) &= 0 \\ \implies x &= \frac{21}{5} \text{ and } \frac{7}{3} \end{aligned}$$

If we take $x < 0$ this becomes:

$$\begin{aligned} 49 - 14x + x^2 &= 4(49 + 28x + 4x^2) \\ 15x^2 + 126x + 147 &= 0 \\ 5x^2 + 42x + 49 &= 0 \\ (5x + 7)(x + 7) &= 0 \\ \implies x &= -\frac{7}{5} \text{ and } -7 \end{aligned}$$

This felt like a lot more work to me, and certainly needed more algebraic manipulation.

(c) From the graphs in the previous two parts we can see that the values $x = -\frac{7}{5}$ and $x = \frac{21}{5}$ do not satisfy $\mathrm{f}(x) = x$, so these are "true" sequences which have period 2.

For $x = -7$ we have $\mathrm{f}(-7) = 7 - 2|-7| = -7$ and so $\mathrm{f}(x) = x$ here, and when $a = -7$ we have a constant sequence (or a sequence with period 1).

For $x = \frac{7}{3}$ we have $\mathrm{f}(\frac{7}{3}) = 7 - 2 \times \frac{7}{3} = \frac{7}{3}$, and so when $a = \frac{7}{3}$ we have another constant sequence.

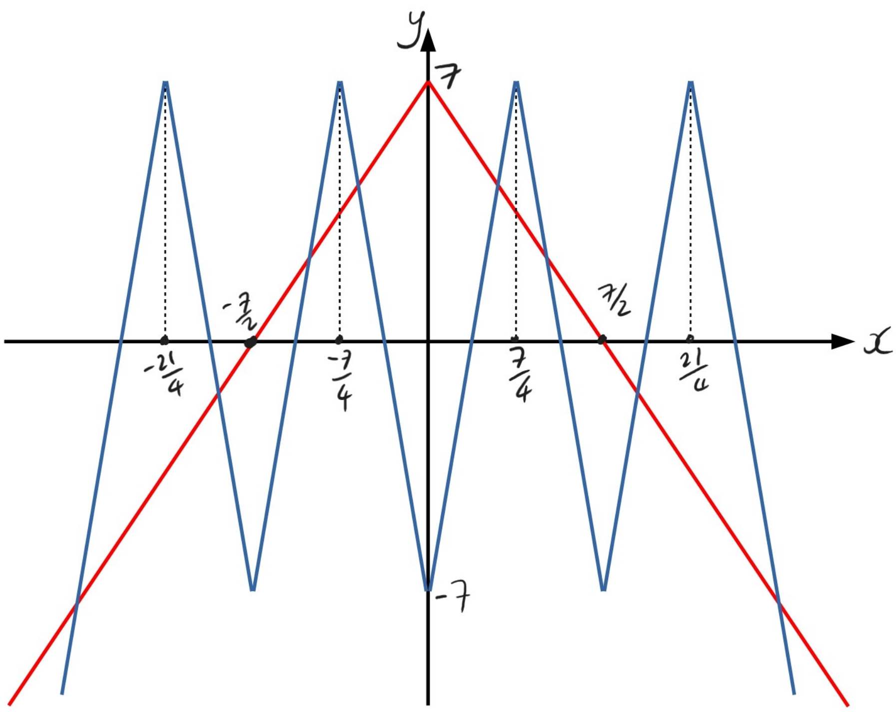
Therefore when $a = -\frac{7}{5}$ and $a = \frac{21}{5}$ we have a sequence with period 2, and when $a = -7$ and $a = \frac{7}{3}$ we have a sequence of period 1.

(d) When $a = \frac{28}{5}$ we have:

$$
\begin{aligned}
u_0 &= \tfrac{28}{5} \\
u_1 &= 7 - 2\left|\tfrac{28}{5}\right| = -\tfrac{21}{5} \\
u_2 &= 7 - 2\left|-\tfrac{21}{5}\right| = -\tfrac{7}{5} \\
u_3 &= 7 - 2\left|-\tfrac{7}{5}\right| = \tfrac{21}{5} \\
u_4 &= 7 - 2\left|\tfrac{21}{5}\right| = -\tfrac{7}{5}
\end{aligned}
$$

So we have $(u_2, u_3, u_4, u_5, \cdots) = \left(-\frac{7}{5}, \frac{21}{5}, -\frac{7}{5}, \frac{21}{5} \cdots\right)$ and neither u_0 nor u_1 equals either of u_2 or u_3.

(ii) **(a)** When we have $\mathrm{f}(\mathrm{f}(x)) = 0$ then we have $\mathrm{f}(\mathrm{f}(\mathrm{f}(x))) = 7$, so at $x = \pm\frac{7}{4}, \pm\frac{21}{4}$ and when $\mathrm{f}(\mathrm{f}(x)) = \pm 7$ then we have $\mathrm{f}(\mathrm{f}(\mathrm{f}(x))) = -7$, so at $x = 0, \pm\frac{7}{2}$.
Using these points we can "join the dots" to get the graph of $y = \mathrm{f}(\mathrm{f}(\mathrm{f}(x)))$.



(b) Sketching $y = x$ onto the graph of $y = \mathrm{f}(\mathrm{f}(\mathrm{f}(x)))$ suggests that we should expect 8 solutions.

As suggested, considerer the sequence starting with $a = 1$:

$$\begin{aligned} u_0 &= 1 \\ u_1 &= 7 - 2 = 5 \\ u_2 &= 7 - 2 \times 5 = -3 \\ u_3 &= 7 - 2 \times 3 = 1 \\ u_4 &= 5 \end{aligned}$$

so we have a sequence of period 3: $(1, 5, -3, 1, 5, -3, \cdots)$.

Starting with $a = -\frac{7}{9}$ we have:

$$\begin{aligned} u_0 &= -\tfrac{7}{9} \\ u_1 &= 7 - 2 \times \tfrac{7}{9} = \tfrac{49}{9} \\ u_2 &= 7 - 2 \times \tfrac{49}{9} = -\tfrac{35}{9} \\ u_3 &= 7 - 2 \times \tfrac{35}{9} = -\tfrac{7}{9} \\ u_4 &= \tfrac{49}{9} \end{aligned}$$

so we have another sequence of period 3: $\left(-\frac{7}{9}, \frac{49}{9}, -\frac{35}{9}, -\frac{7}{9}, \frac{49}{9}, -\frac{35}{9} \cdots\right)$.

We now have 6 solutions to $x = \mathrm{f}(\mathrm{f}(\mathrm{f}(x)))$, i.e. three values each from the two periodic sequences. The other 2 are solutions to $x = \mathrm{f}(x)$, which we found in part **(i)(c)**, i.e. $x = -7$ and $x = -\frac{7}{3}$.
The full set of solutions to $x = \mathrm{f}(\mathrm{f}(\mathrm{f}(x)))$ is

$$x = -7, -\tfrac{35}{9}, -3, -\tfrac{7}{9}, 1, \tfrac{7}{3}, 5, \tfrac{49}{9}$$

(c) We want $u_3, u_4, u_5, \cdots$ to follow one of the two sequences with period 3. To make life easier take $u_3 = 1$ (any of the values in the period 3 sequences from **(ii)(b)** would work, but why would you choose anything other than 1?). Then u_2 satisfies:

$$\begin{aligned} 7 - 2|u_2| &= 1 \\ 2|u_2| &= 6 \\ u_2 &= \pm 3 \end{aligned}$$

As -3 is in the sequence $(1, 5, -3, \cdots)$ already let $u_2 = 3$. Then u_1 satisfies:

$$\begin{aligned} 7 - 2|u_1| &= 3 \\ 2|u_1| &= 4 \\ u_1 &= \pm 2 \end{aligned}$$

Taking $u_1 = 2$ gives:

$$\begin{aligned} 7 - 2|u_0| &= 2 \\ 2|u_0| &= 5 \\ u_0 &= \pm \tfrac{5}{2} \end{aligned}$$

And so taking $a = \frac{5}{2}$ will satisfy the requirements (you only need to find one possible value for a).

Question 3

3 Let $f(x)$ be defined and positive for $x > 0$.

Let a and b be real numbers with $0 < a < b$ and define the points $A = (a, f(a))$ and $B = (b, -f(b))$.

Let $X = (m, 0)$ be the point of intersection of line AB with the x-axis.

(i) Find an expression for m in terms of a, b, $f(a)$ and $f(b)$.

(ii) Show that, if $f(x) = \sqrt{x}$, then $m = \sqrt{ab}$.

Find, in terms of n, a function $f(x)$ such that $m = \dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$.

(iii) Let $g_1(x)$ and $g_2(x)$ be defined and positive for $x > 0$. Let $m = M_1$ when $f(x) = g_1(x)$ and let $m = M_2$ when $f(x) = g_2(x)$.

Show that if $\dfrac{g_1(x)}{g_2(x)}$ is a decreasing function then $M_1 > M_2$.

Hence show that

$$\frac{a+b}{2} > \sqrt{ab} > \frac{2ab}{a+b}.$$

(iv) Let p and c be chosen so that the curve $y = p(c - x)^3$ passes through both A and B. Show that

$$\frac{c-a}{b-c} = \left(\frac{f(a)}{f(b)}\right)^{\frac{1}{3}}$$

and hence determine c in terms of a, b, $f(a)$ and $f(b)$.

Show that if f is a decreasing function, then $c < m$.

Examiner's report

Part **(i)** was generally well done. However, some candidates simply stated the result without showing sufficient working and full credit could not be given. SSP addition: Whilst "Find" in A-level speak means working does not necessarily need to be shown, STEP questions expect justification at all times UNLESS the question clearly says "State" or "Write Down". Sign errors were another common pitfall and usually meant that the accuracy mark could not be awarded.

Part **(ii)** was frequently attempted, though for many candidates it marked the end of their attempt. A common mistake was overlooking the fact that the function $f(x)$ was defined to be positive for $x > 0$, leading to marks being unavailable. Another frequent issue was providing insufficient justification – some candidates simply stated a function without explaining their reasoning or showing it had the required properties, which prevented them from earning full marks.

Part **(iii)** was less frequently attempted, especially the first subpart. Some candidates assumed that the given functions were differentiable and attempted to provide arguments involving derivatives which did not gain credit. The second subpart was generally handled better, though again, a lack of justification was common. A few candidates also attempted alternative methods not involving the previous part, thus ignoring 'hence' in the question.

Part **(iv)** was relatively well done by those who attempted it. The first subpart was accessible to most candidates. The second subpart was more challenging and required careful attention, especially when working with inequalities and avoiding unwarranted assumptions of equality.

Solution

(i) Using $y - y_1 = m(x - x_1)$ the equation of the line AB is given by:

$$y - \mathrm{f}(a) = \frac{-\big[\mathrm{f}(a) + \mathrm{f}(b)\big]}{b - a}(x - a)$$

If this line crosses the x axis at $(m, 0)$ then we have:

$$\begin{aligned}
y - \mathrm{f}(a) &= \frac{-\big[\mathrm{f}(a) + \mathrm{f}(b)\big]}{b - a}(x - a) \\
-\mathrm{f}(a) &= \frac{-\big[\mathrm{f}(a) + \mathrm{f}(b)\big]}{b - a}(m - a) \\
(b - a)\mathrm{f}(a) &= \big[\mathrm{f}(a) + \mathrm{f}(b)\big](m - a) \\
m &= \frac{(b - a)\mathrm{f}(a)}{\mathrm{f}(a) + \mathrm{f}(b)} + a \\
m &= \frac{b\mathrm{f}(a) - \cancel{a\mathrm{f}(a)} + \cancel{a\mathrm{f}(a)} + a\mathrm{f}(b)}{\mathrm{f}(a) + \mathrm{f}(b)} \\
m &= \frac{b\mathrm{f}(a) + a\mathrm{f}(b)}{\mathrm{f}(a) + \mathrm{f}(b)}
\end{aligned}$$

(ii) If $\mathrm{f}(x) = \sqrt{x}$ then we have:

$$\begin{aligned}
m &= \frac{b\mathrm{f}(a) + a\mathrm{f}(b)}{\mathrm{f}(a) + \mathrm{f}(b)} \\
&= \frac{b\sqrt{a} + a\sqrt{b}}{\sqrt{a} + \sqrt{b}} \\
&= \frac{\sqrt{ab}\big[\sqrt{a} + \sqrt{b}\big]}{\sqrt{a} + \sqrt{b}} \\
&= \sqrt{ab}
\end{aligned}$$

Since this is a "show that" question you need to provide enough working to fully justify the given answer. In this case the "show that" means that you can check your answer to part **(i)** before you attempt the rest of the question.

For the next part, lets start by trying $f(x) = x^n$ (this is probably not going to be the right function but might give us some insights into what the right function might be!).

If $f(x) = x^n$ then we have:

$$\begin{aligned} m &= \frac{bf(a) + af(b)}{f(a) + f(b)} \\ &= \frac{b \times a^n + a \times b^n}{a^n + b^n} \end{aligned}$$

Now let's try $f(x) = x^{-n}$:

$$\begin{aligned} m &= \frac{bf(a) + af(b)}{f(a) + f(b)} \\ &= \frac{b \times a^{-n} + a \times b^{-n}}{a^{-n} + b^{-n}} \\ &= \frac{a^n b^n \left[b \times a^{-n} + a \times b^{-n}\right]}{a^n b^n \left[a^{-n} + b^{-n}\right]} \\ &= \frac{b^{n+1} + a^{n+1}}{b^n + a^n} \end{aligned}$$

as required!

(iii) When considering an inequality it's often best to rearrange and try and show whether an equivalent statement is positive or negative. So in this case let's try to prove that $M_1 - M_2 > 0$.

We have:

$$\begin{aligned} M_1 - M_2 &= \frac{bg_1(a) + ag_1(b)}{g_1(a) + g_1(b)} - \frac{bg_2(a) + ag_2(b)}{g_2(a) + g_2(b)} \\ &= \frac{\left[bg_1(a) + ag_1(b)\right]\left[g_2(a) + g_2(b)\right] - \left[bg_2(a) + ag_2(b)\right]\left[g_1(a) + g_1(b)\right]}{\left[g_1(a) + g_1(b)\right]\left[g_2(a) + g_2(b)\right]} \\ &= \frac{bg_1(a)g_2(b) + ag_1(b)g_2(a) - bg_2(a)g_1(b) - ag_2(b)g_1(a)}{\left[g_1(a) + g_1(b)\right]\left[g_2(a) + g_2(b)\right]} \\ &= \frac{g_2(b)g_2(a)}{\left[g_1(a) + g_1(b)\right]\left[g_2(a) + g_2(b)\right]}\left[b\left(\frac{g_1(a)}{g_2(a)} - \frac{g_1(b)}{g_2(b)}\right) + a\left(\frac{g_1(b)}{g_2(b)} - \frac{g_1(a)}{g_2(a)}\right)\right] \\ &= \frac{g_2(b)g_2(a)(b-a)}{\left[g_1(a) + g_1(b)\right]\left[g_2(a) + g_2(b)\right]}\left(\frac{g_1(a)}{g_2(a)} - \frac{g_1(b)}{g_2(b)}\right) \end{aligned}$$

We are told that $g_1(x)$ and $g_2(x)$ are positive (this means strictly greater than 0), and we are told in the stem of the question that $b - a > 0$. If $\frac{g_1(x)}{g_2(x)}$ is decreasing then this means that $\frac{g_1(a)}{g_2(a)} - \frac{g_1(b)}{g_2(b)} > 0$ and hence we have $M_1 - M_2 > 0$ as required.

The next command is **"hence"** so we need to use what we have just done to prove the next part. The middle part of the inequality is m when $\mathrm{f}(x) = \sqrt{x}$.

If we let $\mathrm{g}_1(x) = 1$ then we have $M_1 = \dfrac{a+b}{2}$. If $\mathrm{g}_2(x) = \sqrt{x}$ then $M_2 = \sqrt{ab}$. We know that $\mathrm{g}_1(x)$ and $\mathrm{g}_2(x)$ are positive for $x > 0$, and we also know that $\dfrac{\mathrm{g}_1(x)}{\mathrm{g}_2(x)} = \dfrac{1}{\sqrt{x}}$ is a decreasing function, therefore by we know that $M_1 > M_2$ i.e. $\dfrac{a+b}{2} > \sqrt{ab}$.

Note that this is the AM-GM inequality for two values, but since the command word was "hence" you would have gained no credit for using AM-GM in this case.

For the second part of the inequality, looking back to what we tried in part **(ii)** for inspiration, if we take $\mathrm{g}_3(x) = x$ then this gives $M_3 = \dfrac{ba+ab}{a+b} = \dfrac{2ab}{a+b}$. We also have $\dfrac{\mathrm{g}_2(x)}{\mathrm{g}_3(x)} = \dfrac{\sqrt{x}}{x} = \dfrac{1}{\sqrt{x}}$, which, as before, is a decreasing function. Therefore we have $M_2 > M_3$, i.e. $\sqrt{ab} > \dfrac{2ab}{a+b}$.

Hence we have:

$$\frac{a+b}{2} > \sqrt{ab} > \frac{2ab}{a+b}$$

as required.

(iv) If the curve passes through $A = (a, \mathrm{f}(a))$ and $B = (b, -\mathrm{f}(b))$ then we have:

$$\begin{aligned} \mathrm{f}(a) &= p(c-a)^3 \\ -\mathrm{f}(b) &= p(c-b)^3 \\ \implies \mathrm{f}(b) &= p(b-c)^3 \\ \implies \frac{(c-a)^3}{(b-c)^3} &= \frac{\mathrm{f}(a)}{\mathrm{f}(b)} \\ \implies \frac{c-a}{b-c} &= \left(\frac{\mathrm{f}(a)}{\mathrm{f}(b)}\right)^{\frac{1}{3}} \end{aligned}$$

Rearranging to make c the subject gives:

$$\begin{aligned} c-a &= \left(\frac{\mathrm{f}(a)}{\mathrm{f}(b)}\right)^{\frac{1}{3}}(b-c) \\ c + c \times \left(\frac{\mathrm{f}(a)}{\mathrm{f}(b)}\right)^{\frac{1}{3}} &= b \times \left(\frac{\mathrm{f}(a)}{\mathrm{f}(b)}\right)^{\frac{1}{3}} + a \\ c\,[\mathrm{f}(b)]^{\frac{1}{3}} + c\,[\mathrm{f}(a)]^{\frac{1}{3}} &= b\,[\mathrm{f}(a)]^{\frac{1}{3}} + a\,[\mathrm{f}(b)]^{\frac{1}{3}} \\ c &= \frac{b\,[\mathrm{f}(a)]^{\frac{1}{3}} + a\,[\mathrm{f}(b)]^{\frac{1}{3}}}{[\mathrm{f}(b)]^{\frac{1}{3}} + [\mathrm{f}(a)]^{\frac{1}{3}}} \end{aligned}$$

Considering $c - m$ gives:

$$\begin{aligned}
c - m &= \frac{b\,[\mathrm{f}(a)]^{\frac{1}{3}} + a\,[\mathrm{f}(b)]^{\frac{1}{3}}}{[\mathrm{f}(b)]^{\frac{1}{3}} + [\mathrm{f}(a)]^{\frac{1}{3}}} - \frac{b\mathrm{f}(a) + a\mathrm{f}(b)}{\mathrm{f}(a) + \mathrm{f}(b)} \\
&= \frac{\left(b\,[\mathrm{f}(a)]^{\frac{1}{3}} + a\,[\mathrm{f}(b)]^{\frac{1}{3}}\right)\left(\mathrm{f}(a) + \mathrm{f}(b)\right) - \left(b\mathrm{f}(a) + a\mathrm{f}(b)\right)\left([\mathrm{f}(b)]^{\frac{1}{3}} + [\mathrm{f}(a)]^{\frac{1}{3}}\right)}{\left([\mathrm{f}(b)]^{\frac{1}{3}} + [\mathrm{f}(a)]^{\frac{1}{3}}\right)\left(\mathrm{f}(a) + \mathrm{f}(b)\right)} \\
&= \frac{b\big[\mathrm{f}(a)\big]^{\frac{1}{3}}\mathrm{f}(b) + a\big[\mathrm{f}(b)\big]^{\frac{1}{3}}\mathrm{f}(a) - b\big[\mathrm{f}(b)\big]^{\frac{1}{3}}\mathrm{f}(a) - a\big[\mathrm{f}(a)\big]^{\frac{1}{3}}\mathrm{f}(b)}{\left([\mathrm{f}(b)]^{\frac{1}{3}} + [\mathrm{f}(a)]^{\frac{1}{3}}\right)\left(\mathrm{f}(a) + \mathrm{f}(b)\right)} \\
&= \frac{\big[\mathrm{f}(a)\big]^{\frac{1}{3}}\big[\mathrm{f}(b)\big]^{\frac{1}{3}}\left(b\big[\mathrm{f}(b)\big]^{\frac{2}{3}} + a\big[\mathrm{f}(a)\big]^{\frac{2}{3}} - b\big[\mathrm{f}(a)\big]^{\frac{2}{3}} - a\big[\mathrm{f}(b)\big]^{\frac{2}{3}}\right)}{\left([\mathrm{f}(b)]^{\frac{1}{3}} + [\mathrm{f}(a)]^{\frac{1}{3}}\right)\left(\mathrm{f}(a) + \mathrm{f}(b)\right)} \\
&= \frac{\big[\mathrm{f}(a)\big]^{\frac{1}{3}}\big[\mathrm{f}(b)\big]^{\frac{1}{3}}\left(b - a\right)\left(\big[\mathrm{f}(b)\big]^{\frac{2}{3}} - \big[\mathrm{f}(a)\big]^{\frac{2}{3}}\right)}{\left([\mathrm{f}(b)]^{\frac{1}{3}} + [\mathrm{f}(a)]^{\frac{1}{3}}\right)\left(\mathrm{f}(a) + \mathrm{f}(b)\right)}
\end{aligned}$$

We know that all of the terms are positive apart from possibly $\left(\big[\mathrm{f}(b)\big]^{\frac{2}{3}} - \big[\mathrm{f}(a)\big]^{\frac{2}{3}}\right)$. We know $\mathrm{f}(x)$ is positive but decreasing, and since we are raising to a positive power we know that $\big[\mathrm{f}(x)\big]^{\frac{2}{3}}$ is also decreasing. Hence we have $\big[\mathrm{f}(b)\big]^{\frac{2}{3}} - \big[\mathrm{f}(a)\big]^{\frac{2}{3}} < 0$ and so $c < m$ as required.

Question 4

4 **(i)** x_2 and y_2 are defined in terms of x_1 and y_1 by the equation

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

G_1 is the graph with equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

and G_2 is the graph with equation

$$\frac{\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2}{9} + \frac{\left(-\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2}{4} = 1.$$

Show that, if (x_1, y_1) is a point on G_1, then (x_2, y_2) is a point on G_2.

Show that G_2 is an anti-clockwise rotation of G_1 through $45°$ about the origin.

(ii) **(a)** The matrix

$$\begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix}$$

represents a reflection. Find the line of invariant points of this matrix.

(b) Sketch, on the same axes, the graphs with equations

$$y = 2^x \quad \text{and} \quad 0.8x + 0.6y = 2^{-0.6x+0.8y}.$$

(iii) Sketch, on the same axes, for $0 \leqslant x \leqslant 2\pi$, the graphs with equations

$$y = \sin x \quad \text{and} \quad y = \sin(x - 2y).$$

You should determine the exact co-ordinates of the points on the graph with equation $y = \sin(x - 2y)$ where the tangent is horizontal and those where it is vertical.

Examiner's report

This question was one of the less popular pure questions but still had a good number of attempts. In the question, candidates are led through an example of how to apply the mathematics they know to a new context and then are expected to apply what they have learnt to other problems.

For part **(i)** most candidates knew how to approach the problem but showed insufficient detail in their working for full credit, usually by not making the link (x_1, y_1) on $G_1 \implies \dfrac{x_1^2}{9} + \dfrac{y_1^2}{4} = 1$ when trying to show that (x_2, y_2) on G_2.

A handful did not realise the significance of the indexing on the points and instead tried to show that the equations of the two curves were equivalent.

Almost all candidates recognised that the given matrix was a rotation matrix, but some did not make the link between this and the relationship between the points on the curves clear.

In part **(ii)(a)** it became evident that a number of candidates do not know the difference between a line of invariant points and an invariant line (in the second case points can move under the transformation but must stay on the same line). This meant some candidates did a lot more working than was necessary and often ended up with an extra answer, meaning that they could not gain full credit for this part.

A few candidates used the general form of a reflection matrix in the line $y = \tan\theta$ and a t substitution to find the required line. This method also required candidates to reject one solution, which was usually done by those taking this route.

In part **(ii)(b)** only the most successful candidates showed convincingly that if (x_1, y_1) was on the graph of $y = 2^x$ then $(x_1, y_1) = \begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix} (x_1, y_1)$ was on the other graph, but most realised that the two graphs were reflections of each other and so could make an attempt at the sketch. The most common mistakes here were assuming that the second graph was asymptotic to the y axis and not showing the two graphs intersecting twice.

Attempts were variable for part **(iii)**. A lot of candidates found a matrix connecting points on the two curves, but often had the relationship the 'wrong way around' with $(x_1, y_1) = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} (x_1, y_1)$ rather than the correct version $(x_1, y_1) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} (x_1, y_1)$.

Many candidates could differentiate the implicit equation $y = \sin(x - 2y)$ successfully and some successfully went on to find the points where the tangent was horizontal and vertical. Some who had found the correct transformation matrix could use this to find the points with horizontal tangents but struggled to use a similar argument convincingly for the vertical tangents.

Some candidates successfully set $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\dfrac{\mathrm{d}x}{\mathrm{d}y} = 0$ to get $x - 2y = \dfrac{\pi}{2}$ or $x - 2y = \dfrac{3\pi}{2}$ but were then uncertain how they could use this to find the coordinates of the relevant points.

Many of the candidates who found the points with horizontal or vertical tangents could 'join the dots' to complete the sketch, but some joined them in the wrong order. Many candidates laboured under the misunderstanding that it is not possible for an implicit function to be one-to-many valued which caused a variety of different mistakes.

Solution

(i) If (x_1, y_1) is a point on G_1 then we have:

$$\frac{x_1^2}{9} + \frac{y_1^2}{4} = 1$$

Inverting the relationship at the start of the question gives:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$
$$\implies \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}x_2 + \frac{1}{\sqrt{2}}y_2 \\ -\frac{1}{\sqrt{2}}x_2 + \frac{1}{\sqrt{2}}y_2 \end{pmatrix}$$

Substituting these into the equation for G_1 gives:

$$\frac{x_1^2}{9} + \frac{y_1^2}{4} = 1$$
$$\frac{\left(\frac{1}{\sqrt{2}}x_2 + \frac{1}{\sqrt{2}}y_2\right)^2}{9} + \frac{\left(-\frac{1}{\sqrt{2}}x_2 + \frac{1}{\sqrt{2}}y_2\right)^2}{4} = 1$$

and so if (x_1, y_1) lies on G_1 then (x_2, y_2) lies on G_2.

Alternatively you can substitute for (x_2, y_2) in the left hand side of the equation for G_2 and show that this reduces to $\frac{x_1^2}{9} + \frac{y_1^2}{4}$, and since you know that (x_1, y_1) lies on G_1 you know that this is equal to 1. Care must be taken to show the implication in the correct direction - you do not know that (x_2, y_2) lies on G_2, so you cannot start by stating that $G_2(x_2, y_2) = 1$.

The relationship between points of the curves is given by:

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

Therefore the point (x_2, y_2) is a rotation of 45° anticlockwise of the point (x_1, y_1). Hence the curve G_2 is an anticlockwise rotation of G_1 about the origin by 45°.

(ii) **(a)** The line of invariant points satisfies:

$$\begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\implies -0.6x + 0.8y = x$$
$$\text{and} \quad 0.8x + 0.6y = y$$
$$\implies y = 2x$$

Hence this matrix is a reflection in the line $y = 2x$.

A *line of invariant points* is a line where every point on the line is unaffected by the transformation. If a transformation is a reflection then the line of reflection is a line of invariant points, as every point on the line of reflection is unaffected by the reflection.

An *invariant line* is one where points can move under the transformation but move to another point on the same line. If a transformation is a reflection then all the lines which are perpendicular to the line of reflection are invariant lines. A line of invariant points is a special case of an invariant line.

It is **much** easier to find an equation for a line of invariant points, so do be careful that you are not making life more difficult for yourself!

(b) Consider a point (x_1, y_1) on the first curve and a point (x_2, y_2) on the second curve, so we have:

$$\begin{aligned} y_1 &= 2^{x_1} \\ 0.8x_2 + 0.6y_2 &= 2^{-0.6x_2+0.8y_2} \end{aligned}$$

Therefore we want:

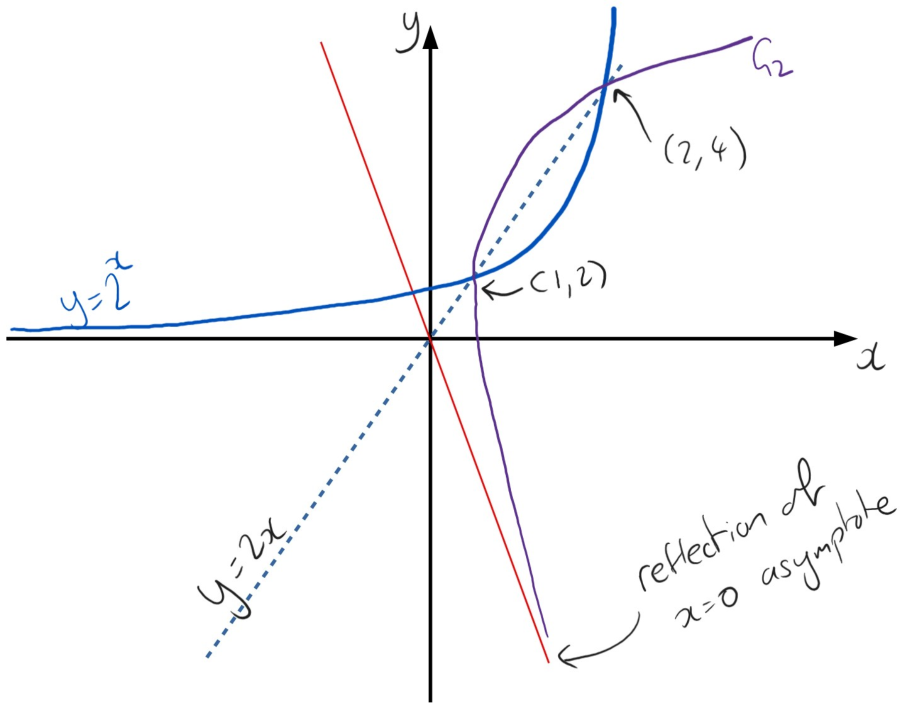
$$\begin{aligned} x_1 &= -0.6x_2 + 0.8y_2 \\ \text{and} \quad y_1 &= 0.8x_2 + 0.6y_2 \\ \text{i.e.} \quad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} &= \begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\ \Longrightarrow \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} &= \begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \end{aligned}$$

Note that since this matrix represents a reflection, the inverse of the matrix is the same as the original matrix!

Hence the second curve is a reflection of $y = 2^x$ in the line $y = 2x$.

Before sketching the graph it would be good to consider whether $y = 2^x$ meets the mirror line at any point. By inspection, we can see that $y = 2^x$ and $y = 2x$ meet at $(1, 2)$ and $(2, 4)$, and by the nature of the graphs there can be no other points of intersection.

This gives us enough information to sketch the graphs.



If you want to find the equation of the new asymptote you can do so by considering what happens to a vector along the x axis under the transformation. We have:

$$\begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix} \begin{pmatrix} p \\ 0 \end{pmatrix} = \begin{pmatrix} -0.6p \\ 0.8p \end{pmatrix}$$

and so the new asymptote is in the direction $\begin{pmatrix} -0.6 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, which has equation $y = -\frac{4}{3}x$.

Finding the equation of the asymptote was not necessary, but realising that the second curve had an oblique asymptote was, and a lot of candidates lost a mark through assuming that the second curve was asymptotic to the negative y axis. Whilst you do not have to draw an accurate graph, you should make sure that your graph of $y = 2x$ has a gradient steeper than 1 to help avoid making this sort of assumption.

(iii) Similarly to before, let (x_1, y_1) be a point on the first curve and let (x_2, y_2) be a point on the second curve, so we have:

$$\begin{aligned} y_1 &= \sin x_1 \\ y_2 &= \sin(x_2 - 2y_2) \\ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} &= \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\ \implies \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \end{aligned}$$

This matrix represents something called a *shear* of $y = \sin x$ parallel to the x axis. However it is not necessary to know this to sketch the graphs, and I find this sort of transformation a little tricky to visualise anyway so this knowledge does not help me much! The method I use in these cases is consider what happens to some of the key points of $y = \sin x$.

Where $y = \sin x$ crosses the x axis, these points map to:

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k\pi \\ 0 \end{pmatrix} = \begin{pmatrix} k\pi \\ 0 \end{pmatrix}$$

and so the two curves cross the x axis in the same places.

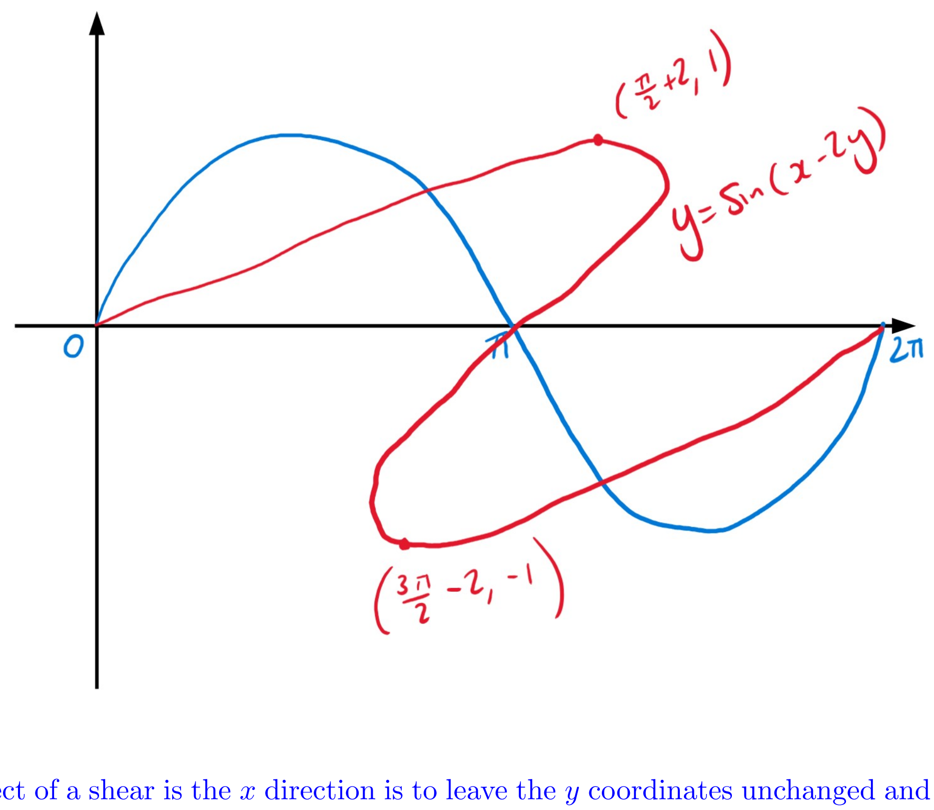
The maximum of $y = \sin x$, i.e. $\left(\frac{1}{2}\pi, 1\right)$, maps to:

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\pi \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\pi + 2 \\ 1 \end{pmatrix}$$

and the minimum point $\left(\frac{3}{2}\pi, -1\right)$ maps to:

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2}\pi \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2}\pi - 2 \\ -1 \end{pmatrix}$$

Noting that $\frac{1}{2}\pi + 2 > \pi$ and $\frac{3}{2}\pi - 2 < \pi$ we have enough information to sketch the basic shape of the curve.



The affect of a shear is the x direction is to leave the y coordinates unchanged and stretch the x coordinates by an amount which is proportional to the y coordinate. This affect is slightly complicated when the y coordinate is negative which is why I find it is safer to work out what the transformation does to a few special points first.

Note the the equation $y = \sin(x-2y)$ is an *implicit* one, i.e. it doesn't have y (or x) on it's own on one side. This means that there can be more than one y value for a single x value, which some candidates seemed reluctant to draw. Probably the easiest implicit curve to consider is $x^2 + y^2 = 1$, and no-one seems to have a problem with this one being multi-valued!

The question also asks us to determine the points where the tangent is vertical and where it is horizontal. Since the shear is in the x direction then the points where the tangent is horizontal are the ones which come from the points on $y = \sin x$ where the tangent is horizontal, i.e. the two points $\left(\frac{\pi}{2} + 2, 1\right)$ and $\left(\frac{3\pi}{2} - 2, -1\right)$ found earlier.

For the points where the tangent is vertical we have:

$$\begin{aligned} y &= \sin(x-2y) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \left(1 - 2\frac{\mathrm{d}y}{\mathrm{d}x}\right)\cos(x-2y) \\ \implies \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\cos(x-2y)}{1+2\cos(x-2y)} \end{aligned}$$

Where the tangent is vertical we have:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}y} &= 0 \\ \implies 1 + 2\cos(x-2y) &= 0 \\ \cos(x-2y) &= -\frac{1}{2} \\ \implies x - 2y &= \frac{2\pi}{3}, \frac{4\pi}{3}, \cdots \end{aligned}$$

If $x - 2y = \frac{2\pi}{3}$ then we have $y = \sin(x-2y) = \frac{\sqrt{3}}{2}$ and so:

$$\begin{aligned} x - 2y &= \frac{2\pi}{3} \\ \implies x - \sqrt{3} &= \frac{2\pi}{3} \\ x &= \frac{2\pi}{3} + \sqrt{3} \end{aligned}$$

and so one of the points where the tangent is vertical is at $\left(\frac{2\pi}{3} + \sqrt{3}, \frac{\sqrt{3}}{2}\right)$.

Similarly if $x - 2y = \frac{4\pi}{3}$ then we have $y = \sin(x-2y) = -\frac{\sqrt{3}}{2}$ and by using a similar method we have the other point where the tangent is vertical being at $\left(\frac{4\pi}{3} - \sqrt{3}, -\frac{\sqrt{3}}{2}\right)$.

Looking at the graph that we have drawn for this question these results seem plausible!

You an also use the gradient method to find the points where the gradient is horizontal. We have:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \implies \cos(x-2y) &= 0 \\ \implies x - 2y &= \frac{\pi}{2}, \frac{3\pi}{2}, \cdots \end{aligned}$$

Then following the same method as above will give the points $\left(\frac{\pi}{2} + 2, 1\right)$ and $\left(\frac{3\pi}{2} - 2, -1\right)$.

Question 5

5 Three points, A, B and C, lie in a horizontal plane, but are not collinear. The point O lies above the plane.

Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

P is a point with $\overrightarrow{OP} = \alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}$, where α, β and γ are all positive and $\alpha+\beta+\gamma < 1$.

Let $k = 1 - (\alpha + \beta + \gamma)$.

(i) The point L is on OA, the point X is on BC and LX passes through P.

Determine $\overrightarrow{OX}$ in terms of β, γ, $\mathbf{b}$ and $\mathbf{c}$ and show that $\overrightarrow{OL} = \dfrac{\alpha}{k+\alpha}\mathbf{a}$.

(ii) Let M and Y be the unique pair of points on OB and CA respectively such that MY passes through P, and let N and Z be the unique pair of points on OC and AB respectively such that NZ passes through P.

Show that the plane LMN is also horizontal if and only if OP intersects plane ABC at the point G, where $\overrightarrow{OG} \equiv \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$. Where do points X, Y and Z lie in this case?

(iii) State what the condition $\alpha+\beta+\gamma < 1$ tells you about the position of P relative to the tetrahedron $OABC$.

Examiner's report

This question was the least popular pure question by a large margin, and of the attempts made less than half were 'substantial' attempts.

As is often the case with vector questions, a carefully drawn diagram can be very helpful in selecting an appropriate method for solving the question and the most successful candidates made good use of this.

Various methods were used in part **(i)**, but mostly these involved finding vector equations of relevant lines and manipulating these to show the required results. Candidates should be aware that questions on the STEP papers need enough justification to fully support their solutions. Many candidates lost accuracy marks through their argument not being convincing enough or lacking some details.

Part **(ii)** had some very good solutions, but many candidates found it difficult to understand what it means for LMN to be horizontal. A clear diagram here would have helped candidates to find a solution method.

Some candidates tried to do both directions of the 'if and only if' in one go. They usually did not gain full marks here, either because they did not link one pair of statements with an if and only if symbol or because they did not appreciate that one step needed a different approach for each direction of implication. It is always 'safer' to approach each direction of implication separately.

The most common issues were not using $k \neq 0$ when justifying $\dfrac{\alpha}{k+\alpha} = \dfrac{\beta}{k+\beta} \implies \alpha = \beta$, or for not convincingly explaining why $\alpha = \beta = \gamma$ means that LMN was horizontal.

Part **(iii)** required a one-line answer, and some candidates who had taken the time to read the whole question successfully answered this part even if they had not answered the previous parts. Some candidates confused 'positive' with 'non-negative' and stated that point P could be inside or on the faces of the tetrahedron.

Solution

You might like to have a look at STEP 1 1987 Question 9 for a more on points of the form $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$.

For a lot of vector questions a diagram will help you work out how to solve the problem. The first diagram I drew (and in fact used to solve the whole question) had O **below** the plane ABC, though that doesn't affect any of the working in this case!

The fact that ABC is horizontal does not matter, it could be vertical or at any angle and the solution method would not change.

There is a slight notational subtlety that should be mentioned, which is that $\overrightarrow{OX}$ represents the vector between points O and X and OX is the (infinite) line that passes through O and X.

(i) Since L is on OA we have $\overrightarrow{OL} = \delta\mathbf{a}$. Since X is on BC we have:

$$\begin{aligned}\overrightarrow{OX} &= \overrightarrow{OB} + \lambda\overrightarrow{BC}\\ &= \overrightarrow{OB} + \lambda\left(-\overrightarrow{OB} + \overrightarrow{OC}\right)\\ &= (1-\lambda)\overrightarrow{OB} + \lambda\overrightarrow{OC}\\ &= (1-\lambda)\mathbf{b} + \lambda\mathbf{c}\end{aligned}$$

We also know that P is on LX, so in a similar way we have:

$$\begin{aligned}\overrightarrow{OP} &= (1-\mu)\overrightarrow{OL} + \mu\overrightarrow{OX}\\ &= (1-\mu)\delta\mathbf{a} + \mu\big[(1-\lambda)\mathbf{b} + \lambda\mathbf{c}\big]\end{aligned}$$

Since A, B and C are not collinear then we know that we cannot write one of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in terms of the other two. This means that the coefficients in the two expressions $\overrightarrow{OP} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$ and $\overrightarrow{OP} = (1-\mu)\delta\mathbf{a} + \mu\big[(1-\lambda)\mathbf{b} + \lambda\mathbf{c}\big]$ must be the same. Equating coefficients gives:

$$\alpha = (1-\mu)\delta \tag{1}$$
$$\beta = \mu(1-\lambda) \tag{2}$$
$$\gamma = \mu\lambda \tag{3}$$

Adding (2) and (3) gives:

$$\mu = \beta + \gamma$$
$$\lambda = \frac{\gamma}{\mu} = \frac{\gamma}{\beta + \gamma}$$

Substituting this into the expression for $\overrightarrow{OX}$ gives:

$$\begin{aligned}\overrightarrow{OX} &= (1-\lambda)\mathbf{b} + \lambda\mathbf{c} \\ &= \left(1 - \frac{\gamma}{\beta+\gamma}\right)\mathbf{b} + \frac{\gamma}{\beta+\gamma}\mathbf{c} \\ &= \frac{\beta}{\beta+\gamma}\mathbf{b} + \frac{\gamma}{\beta+\gamma}\mathbf{c}\end{aligned}$$

Substituting for μ into equation (1) gives:

$$\begin{aligned}\delta &= \frac{\alpha}{1-\mu} \\ &= \frac{\alpha}{1-\beta-\gamma} \\ &= \frac{\alpha}{k+\alpha}\end{aligned}$$

and so we have $\overrightarrow{OL} = \dfrac{\alpha}{k+\alpha}\mathbf{a}$ as required.

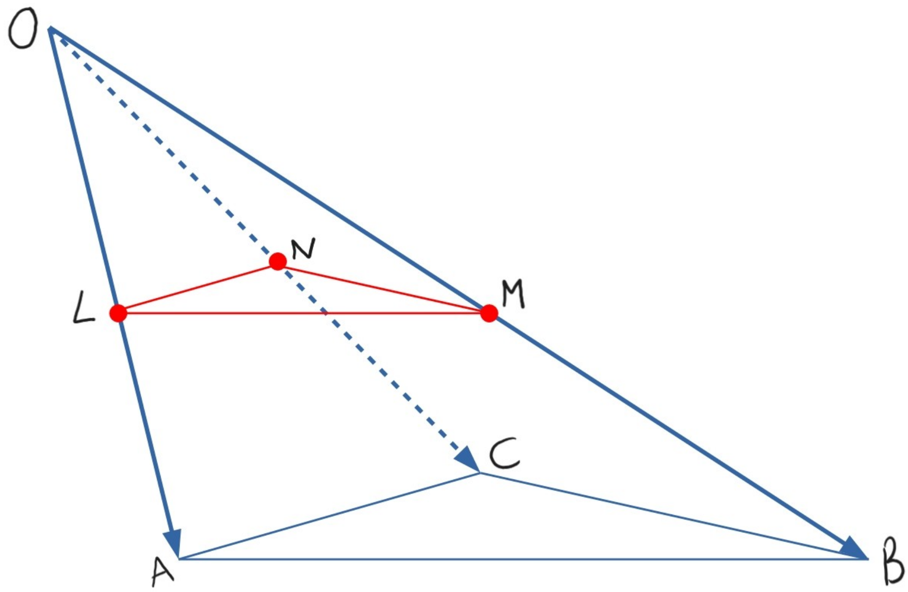
Note that the command words in this part were "**Determine**" and "**Show That**". This means that you need to show enough working to fully justify your answer and the flow of logic through the solution should be clear. However it is always advisable to show full justification for all your solutions, unless the question makes it clear that you don't need to by stating "Write Down" or "State".

There are other ways you could approach this, such as using $\overrightarrow{OX} = \overrightarrow{OL} + \overrightarrow{LX} = \overrightarrow{OL} + \lambda\overrightarrow{LP}$, or equating two different expressions for $\overrightarrow{LX}$. But in all cases you are finding two different expressions for one vector and eqauting the coefficients of **a**, **b** and **c**.

(ii) In the previous part we showed that $\overrightarrow{OX} = \frac{\beta}{\beta+\gamma}\mathbf{b} + \frac{\gamma}{\beta+\gamma}\mathbf{c}$ and that $\overrightarrow{OL} = \dfrac{\alpha}{k+\alpha}\mathbf{a}$. By symmetry we also have:

$$\begin{aligned}\overrightarrow{OY} &= \frac{\alpha}{\alpha+\gamma}\mathbf{a} + \frac{\gamma}{\alpha+\gamma}\mathbf{c} \\ \overrightarrow{OM} &= \frac{\beta}{k+\beta}\mathbf{a} \\ \overrightarrow{OZ} &= \frac{\alpha}{\alpha+\gamma}\mathbf{a} + \frac{\beta}{\beta+\gamma}\mathbf{b} \\ \overrightarrow{ON} &= \frac{\gamma}{k+\gamma}\mathbf{a}\end{aligned}$$

For this part a diagram is very helpful:



From this we can see that plane LMN is parallel to plane ABC (and so is horizontal) if $\overrightarrow{LM}$ is parallel to $\overrightarrow{AB}$ and $\overrightarrow{LN}$ is parallel to $\overrightarrow{AC}$ (You can consider any two pairs of vectors here).

A little bit of care is needed with the "if and only if", and it is safest to consider each direction separately.

If LMN is horizontal then we know that $\overrightarrow{LM}$ is parallel to $\overrightarrow{AB}$. We have:

$$\begin{aligned}\overrightarrow{LM} &= \overrightarrow{OM} - \overrightarrow{OL} \\ &= \frac{\beta}{k+\beta}\mathbf{b} - \frac{\alpha}{k+\alpha}\mathbf{a} \\ &= \lambda(\mathbf{b} - \mathbf{a})\end{aligned}$$

Where the last line comes from $\overrightarrow{LM}$ is parallel to $\overrightarrow{AB}$.

This means that we have:

$$\begin{aligned}\frac{\beta}{k+\beta} &= \frac{\alpha}{k+\alpha} \\ \beta(k+\alpha) &= \alpha(k+\beta) \\ k\beta &= k\alpha \\ k \neq 0 \implies \beta &= \alpha\end{aligned}$$

Similarly if $\overrightarrow{LN}$ is parallel to $\overrightarrow{AC}$ then we have $\alpha = \gamma$. Therefore we have $\alpha = \beta = \gamma$. Therefore we have $\overrightarrow{OP} = \alpha(\mathbf{a} + \mathbf{b} + \mathbf{c})$.

The equation of the plane ABC is $r = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a}) = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. Where $\overrightarrow{OP}$ meets the plane we have $\lambda = \mu = 1 - \mu - \lambda \implies \mu = \lambda = \frac{1}{3}$. Therefore point G satisfies $\overrightarrow{OG} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$.

This means that we have shown that LMN horizontal implies that OP intersects plane ABC at the point G where $\overrightarrow{OG} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$.

For the other direction of implication, if we know that OP passes through point G then we must have:

$$\overrightarrow{OP} = \lambda\overrightarrow{OG}$$
$$\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = \frac{1}{3}\lambda(\mathbf{a} + \mathbf{b} + \mathbf{c})$$
$$\implies \alpha = \beta = \gamma$$

Therefore we have $\overrightarrow{LM} = \frac{\beta}{k+\beta}\mathbf{b} - \frac{\alpha}{k+\alpha}\mathbf{a} = \frac{\alpha}{k+\alpha}(\mathbf{b} - \mathbf{a})$ and so $\overrightarrow{LM}$is parallel to $\overrightarrow{AB}$. Similarly we have $\overrightarrow{LN}$is parallel to $\overrightarrow{AC}$ and so plane LMN is parallel to plane ABC.

Note that there are two steps which are slightly different in the two directions of implication. In the first direction we have to use the fact that $k \neq 0$ to conclude that $\alpha = \beta = \gamma$ — this step does not appear in the second direction. Also, in the first direction we need to show that the point G satisfies $\overrightarrow{OG} = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$, this also does not have to happen in the second direction.

When $\beta = \gamma$ we have $\overrightarrow{OX} = \frac{\beta}{2\beta}\mathbf{b} + \frac{\beta}{2\beta}\mathbf{c} = \frac{1}{2}(\mathbf{b}+\mathbf{c})$ and so X is the midpoint of BC. Similarly Y is the midpoint of AC and Z is the midpoint of AB.

There are other ways of showing that plane LMN is parallel to plane ABC including:

- Show that L, M and N are the same proportion of the length along OA, OB and OC, i.e. $|OL| = \lambda|OA|$, $|OM| = \lambda|OB|$ and $|ON| = \lambda|OC|$
- The plane ABC has directions $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$, so plane LMN is parallel if we can write $\overrightarrow{LM} = r(\mathbf{b} - \mathbf{a}) + s(\mathbf{c} - \mathbf{a})$ and similarly for $\overrightarrow{LN}$.
- Considering the normal to plane ABC, $\mathbf{n} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$ and the normal to plane LMN.

(iii) The fact that $\alpha + \beta + \gamma < 1$, combined with the fact that α, β and γ are all positive, means that P lies inside the tetrahedron $OABC$.

In STEP questions "positive" means strictly greater than zero. If a value is greater than or equal to zero then we call this "non-negative".

Note that if we had $\alpha = 0$ then point P would be on face OBC of the tetrahedron.

Note here that the question said "State" which implies that no working was necessary, and there was only one mark allocated to this part.

Question 6

6 **(i)** Let a, b and c be three non-zero complex numbers with the properties $a+b+c = 0$ and $a^2 + b^2 + c^2 = 0$.

Show that a, b and c cannot all be real.

Show further that a, b and c all have the same modulus.

(ii) Show that it is not possible to find three non-zero complex numbers a, b and c with the properties $a + b + c = 0$ and $a^3 + b^3 + c^3 = 0$.

(iii) Show that if any four non-zero complex numbers a, b, c and d have the properties $a + b + c + d = 0$ and $a^3 + b^3 + c^3 + d^3 = 0$, then at least two of them must have the same modulus.

(iv) Show, by taking $c = 1$, $d = -2$ and $e = 3$ that it is possible to find five real numbers a, b, c, d and e with distinct magnitudes and with the properties $a + b + c + d + e = 0$ and $a^3 + b^3 + c^3 + d^3 + e^3 = 0$.

Examiner's report

Question **6** proved to be quite challenging for many candidates, with a significant number scoring fewer than 5 marks and only attempting part **(i)**, or part **(i)** and part **(ii)**.

In part **(i)**, the first mark was easily earned for strict inequalities or stating the only solution is the zero solution but was not earned if it was stated that "the square of any real is positive" rather than any non-zero real. For the rest of the question, it was very common for candidates to attempt to consider each of a, b, c in the form $x + y\mathrm{i}$, and then substitute in to obtain four equations in six variables. Those that tried this invariably made no progress. While it is possible to answer the question using real and imaginary parts, it requires far more work and so no credit was awarded for just writing down these four equations. Those who left the algebra in terms of a, b, c or used the roots of a quadratic tended to answer this part well.

Part **(ii)** also saw attempts to split a, b, c into real and imaginary parts. This saw no further progress, or credit. The most common way that this question was answered was by writing down an identity relating the sum of cubes to the cube of the sum. This identity could be written in several equivalent forms, and saw many errors in the coefficients and signs, for which candidates were penalised accuracy marks.

Careful thought about presentation was required before commencing the algebraic manipulation to part **(iii)** to avoid introducing sign and arithmetic errors. Complicated identities were common and often contained errors. Establishing that $abc + bcd + acd + abd = 0$ was a common approach and led to considering the roots of a quartic.

Part **(iv)** was generally well answered. Most candidates that attempted it were able to identify a quadratic and solve it. Several candidates that could not solve some of parts **(i)**, **(ii)**, **(iii)** skipped straight to this part and picked up some marks. This is good general exam practice and STEP candidates should remember that subsequent parts of a question can often still be answered even if an early part seems challenging.

Solution

(i) We are told that a, b and c are non-zero. If a is real and not equal to zero then we know that $a^2 > 0$, and similarly for b and c. Therefore if all of a, b and c are real and non-zero we have $a^2+b^2+c^2 > 0$ which is a contradiction. Therefore at least one of a, b and c is not real. Note that since $a+b+c=0$ then this means that at least two are non-real.

We have $a = -b-c$, so $a^2 = (b+c)^2$. This means that we have:

$$\begin{aligned} a^2+b^2+c^2 &= 0 \\ a^2 + (b+c)^2 - 2bc &= 0 \\ a^2 + a^2 &= 2bc \\ a^2 &= bc \end{aligned}$$

The second line here is an example of "adding zero creatively" - By adding in "$2bc - 2bc$" we can complete a square and make things look nicer.

By symmetry we also have $b^2 = ac$ and $c^2 = ab$. Therefore we have $a^3 = b^3 = c^3 = abc$ and so $|a| = |b| = |c|$.

Alternate method. Using $a^2 = (b+c)^2$ we have:

$$\begin{aligned} a^2+b^2+c^2 &= 0 \\ b^2+2bc+c^2+b^2+c^2 &= 0 \\ b^2+bc+c^2 &= 0 \\ \implies b &= \frac{-c \pm \sqrt{c^2-4c^2}}{2} \\ b &= c\left(\frac{-1 \pm \mathrm{i}\sqrt{3}}{2}\right) \end{aligned}$$

Note that $\left|\dfrac{-1 \pm \mathrm{i}\sqrt{3}}{2}\right| = \dfrac{1+3}{4} = 1$, hence we have $|b| = |c|$, and by symmetry we have $|a| = |b| = |c|$.

(ii) Using $a = -b-c$ gives:

$$\begin{aligned} a^3+b^3+c^3 &= 0 \\ (-b-c)^3+b^3+c^3 &= 0 \\ -b^3-3b^2c-3bc^2-c^3+b^3+c^3 &= 0 \\ 3bc(b+c) &= 0 \\ \implies b+c &= 0 \quad \text{as } b, c \neq 0 \end{aligned}$$

But if $b+c=0$ then this implies that $a=0$.

Therefore it is not possible to find three non-zero numbers that satisfy the equations.

(iii) Using the sum of two cubes factorisation we have:

$$
\begin{aligned}
a^3 + b^3 + c^3 + d^3 &= 0 \\
(a+b)\left(a^2 - ab + b^2\right) + (c+d)\left(c^2 - cd + d^2\right) &= 0 \\
(a+b)\left(a^2 - ab + b^2\right) + (-a-b)\left(c^2 - cd + d^2\right) &= 0 \\
(a+b)\Big(a^2 - ab + b^2 - c^2 + cd - d^2\Big) &= 0
\end{aligned}
$$

So either $a = -b$ so they have the same modulus, or:

$$
\begin{aligned}
a^2 - ab + b^2 &= c^2 - cd + d^2 \\
(a+b)^2 - 3ab &= (c+d)^2 - 3cd \\
3ab &= 3cd \quad \text{as } (a+b)^2 = (c+d)^2
\end{aligned}
$$

Hence we have $a = \dfrac{cd}{b}$ and so:

$$
\begin{aligned}
a + b + c + d &= 0 \\
\frac{cd}{b} + b + c + d &= 0 \\
cd + b^2 + cb + db &= 0 \\
(b+c)(b+d) &= 0
\end{aligned}
$$

So either $b = -c$ or $b = -d$ and so two numbers have the same modulus.

(iv) If $c = 1, d = -2$ and $e = 3$ then we have:

$$
\begin{aligned}
a + b + 2 &= 0 \\
a^3 + b^3 + 20 &= 0 \\
a^3 + (-a-2)^3 + 20 &= 0 \\
\cancel{a^3} - \cancel{a^3} - 6a^2 - 12a - 8 + 20 &= 0 \\
a^2 + 2a - 2 &= 0 \\
a &= \frac{-2 \pm \sqrt{4+8}}{2} \\
a &= -1 \pm \sqrt{3}
\end{aligned}
$$

Therefore if we take $a = -1 + \sqrt{3}$, $b = -1 - \sqrt{3}$, $c = 1, d = -2$ and $e = 3$ then we have five real numbers with distinct magnitudes that satisfy the given equations.

Question 7

7 Let $\mathrm{f}(x) = \sqrt{x^2+1} - x$.

(i) Using a binomial series, or otherwise, show that, for large $|x|$, $\sqrt{x^2+1} \approx |x| + \dfrac{1}{2|x|}$.

Sketch the graph $y = \mathrm{f}(x)$.

(ii) Let $\mathrm{g}(x) = \tan^{-1} \mathrm{f}(x)$ and, for $x \neq 0$, let $\mathrm{k}(x) = \frac{1}{2} \tan^{-1} \dfrac{1}{x}$.

(a) Show that $\mathrm{g}(x) + \mathrm{g}(-x) = \frac{1}{2}\pi$.

(b) Show that $\mathrm{k}(x) + \mathrm{k}(-x) = 0$.

(c) Show that $\tan \mathrm{k}(x) = \tan \mathrm{g}(x)$ for $x > 0$.

(d) Sketch the graphs $y = \mathrm{g}(x)$ and $y = \mathrm{k}(x)$ on the same axes.

(e) Evaluate $\displaystyle\int_0^1 \mathrm{k}(x)\,\mathrm{d}x$ and hence write down the value of $\displaystyle\int_{-1}^0 \mathrm{g}(x)\,\mathrm{d}x$.

Examiner's report

In part **(i)**, some candidates tried to expand $\sqrt{1+x^2}$ as a series in increasing powers of x^2, not appreciating that they needed $|x|$ to be small for such an expansion to be valid.

A lot of candidates used the expansion to correctly identify the asymptotes of $\mathrm{f}(x)$.

In part **(ii)(a)**, the most common approach was to consider $\tan\big(\mathrm{g}(x)+\mathrm{g}(-x)\big)$, which, using the tan double angle formula, "$= \infty$". Only a small number justified their answer by using the positivity of $\mathrm{f}(x)$ to get $0 < \mathrm{g}(x) + \mathrm{g}(-x) < \pi$. Many candidates then simply stated the answer, or wrote $\mathrm{g}(x) + \mathrm{g}(-x) = \tan^{-1} \infty$, stating that this is $\dfrac{\pi}{2}$. A common theme in this question was a lack of consideration of ranges/domains of the trigonometric functions which meant there were marks that were unavailable.

Part **(ii)(b)** was done well in general, with many candidates knowing that $y = \arctan x$ is an odd function. Some overcomplicated it, using the double angle formula again, and not gaining a mark for justifying the range of $\mathrm{k}(x)$, i.e. $\tan\big(\mathrm{k}(x)+\mathrm{k}(-x)\big) = 0 \not\Rightarrow \mathrm{k}(x) + \mathrm{k}(-x) = 0$ in general.

In part **(ii)(c)**, the most common approach was again to use the tan double angle formula, realising that $\tan\big(2\mathrm{k}(x)\big) = x^{-1}$, and arriving at a quadratic for $\tan\big(\mathrm{k}(x)\big)$. Marks were again unavailable for those candidates that either did not attempt to solve the quadratic or did solve for the two roots but then not explaining why $\tan\big(\mathrm{k}(x)\big) = \mathrm{f}(x)$ was the correct root to choose.

There were also some nice geometric arguments for part **(ii)(c)**, drawing a right-angled triangle with angle $2\mathrm{k}(x)$, then bisecting the angle and finding the side lengths of the smaller right-angled triangle with angle $2\mathrm{k}(x)$ to find $\tan\big(\mathrm{k}(x)\big)$.

The sketches in part **(ii)(d)** were good in general, although some candidates' sketches contradicted the relations for $\mathrm{g}(x)$ and $\mathrm{k}(x)$, given in the question, for example sketching $\mathrm{k}(x)$ as an even function, or not using $\tan\big(\mathrm{k}(x)\big) = \tan\big(\mathrm{g}(x)\big)$ for $x > 0$

Part **(ii)(e)** was done well by candidates who attempted it. Some overcomplicated the integral by changing variables, but the majority realised they could integrate by parts directly. For the last part, candidates either used their sketches to find the right area or integrated the relation in part **(a)** directly using part **(c)**.

Solution

(i) Note that a square root always returns a positive value, for example $\sqrt{(-2)^2} = \sqrt{4} = 2$. In general we gave $\sqrt{x^2} = |x|$.

We have:

$$\begin{aligned}
(x^2+1)^{\frac{1}{2}} &= \sqrt{x^2}\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}} \\
&= |x|\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}} \\
&= |x|\left(1+\frac{1}{2}\times\frac{1}{x^2}+\cdots\right) \\
&\approx |x|\left(1+\frac{1}{2|x|^2}\right) \qquad \text{for large } |x| \\
\implies \sqrt{x^2+1} &\approx |x| + \frac{1}{2|x|}
\end{aligned}$$

So as $x \to +\infty$ we have $\mathrm{f}(x) \approx |x| + \dfrac{1}{2|x|} - x = \dfrac{1}{2x}$ and the graph tends to the x axis in this direction.

As $x \to -\infty$ we have $\mathrm{f}(x) \approx |x| + \dfrac{1}{2|x|} - x = \dfrac{-1}{2x} - 2x$, and so the graph tends to $y = -2x$ as $x \to -\infty$.

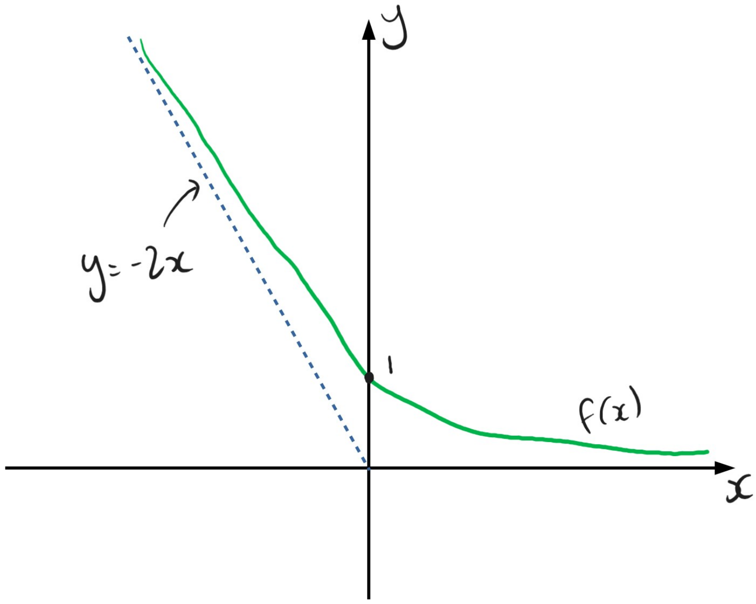
The gradient of the graph is given by:

$$\begin{aligned}
\mathrm{f}'(x) &= \tfrac{1}{2}\times 2x \times \left(x^2+1\right)^{-\frac{1}{2}} - 1 \\
&= \frac{x}{\sqrt{x^2+1}} - 1
\end{aligned}$$

This means that the gradient is always negative. Note also that as $x \to \infty$, $\mathrm{f}'(x) \to 0$ and as $x \to -\infty$. $\mathrm{f}'(x) \to -2$ which matches our asymptotes.

When $x = 0$ we have $y = 1$.

This gives us enough information to sketch the graph:



(ii) **(a)** Consider $\mathrm{f}(x)\mathrm{f}(-x)$:

$$\begin{aligned}\mathrm{f}(x)\mathrm{f}(-x) &= \left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)\\ &= x^2+1-x^2\\ &= 1\end{aligned}$$

We also have $\tan\left(\mathrm{g}(x)\right) = \tan\left(\tan^{-1}\left(\mathrm{f}(x)\right)\right) = \mathrm{f}(x)$.

Note that $\tan\left(\tan^{-1}(\theta)\right) \equiv \theta$ BUT $\tan^{-1}\left(\tan(\theta)\right)$ is NOT necessarily equal to θ. You can use a sketch of $y = \tan x$ to convince yourself of this.

Therefore we have:

$$\begin{aligned}\tan\left(\mathrm{g}(x)\right)\tan\left(\mathrm{g}(-x)\right) &= \mathrm{f}(x)\mathrm{f}(-x) = 1\\ \implies \sin\left(\mathrm{g}(x)\right)\sin\left(\mathrm{g}(-x)\right) &= \cos\left(\mathrm{g}(x)\right)\cos\left(\mathrm{g}(-x)\right)\\ \implies \cos\left(\mathrm{g}(x)\right)\cos\left(\mathrm{g}(-x)\right) - \sin\left(\mathrm{g}(x)\right)\sin\left(\mathrm{g}(-x)\right) &= 0\\ \cos\left(\mathrm{g}(x)+\mathrm{g}(-x)\right) &= 0\\ \implies \mathrm{g}(x)+\mathrm{g}(-x) &= \frac{\pi}{2}+m\pi\end{aligned}$$

We know that $\mathrm{f}(x) > 0$ for all x, and so $\mathrm{g}(x) = \tan^{-1}\left(\mathrm{f}(x)\right) \in (0, \frac{\pi}{2})$, and so $\mathrm{g}(x) + \mathrm{g}(-x) \in (0, \pi)$.
Therefore we have $\mathrm{g}(x) + \mathrm{g}(-x) = \frac{\pi}{2}$.

First time I did this question I considered $\tan\left(\mathrm{g}(x)+\mathrm{g}(-x)\right)$ and found that this was undefined. With some vigorous hand waving I said that this mean that $\mathrm{g}(x)+\mathrm{g}(-x) = \frac{\pi}{2}$, but this didn't feel like an entirely satisfactory method.

(b) We have:

$$\begin{aligned}
\mathrm{k}(x) + \mathrm{k}(-x) &= \tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{x}\right) + \tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{-x}\right) \\
&= \tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{x}\right) - \tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{x}\right) \\
&= 0
\end{aligned}$$

as required.

(c) When $x > 0$ we have $\dfrac{1}{x} = \tan\big(2\mathrm{k}(x)\big)$. Using the double tan formula we have:

$$\begin{aligned}
\frac{1}{x} &= \tan\big(2\mathrm{k}(x)\big) \\
&= \frac{2\tan\big(\mathrm{k}(x)\big)}{1 - \tan^2\big(\mathrm{k}(x)\big)} \\
\implies 1 - \tan^2\big(\mathrm{k}(x)\big) &= 2x\tan\big(\mathrm{k}(x)\big) \\
0 &= \tan^2\big(\mathrm{k}(x)\big) + 2x\tan\big(\mathrm{k}(x)\big) - 1 \\
\tan\big(\mathrm{k}(x)\big) &= \frac{-2x \pm \sqrt{4x^2+4}}{2} \\
\tan\big(\mathrm{k}(x)\big) &= -x \pm \sqrt{x^2+1}
\end{aligned}$$

Since $x > 0$ we know $\mathrm{k}(x) \in \left(0, \frac{\pi}{4}\right)$ and so $\tan\big(\mathrm{k}(x)\big) > 0$ and we have:

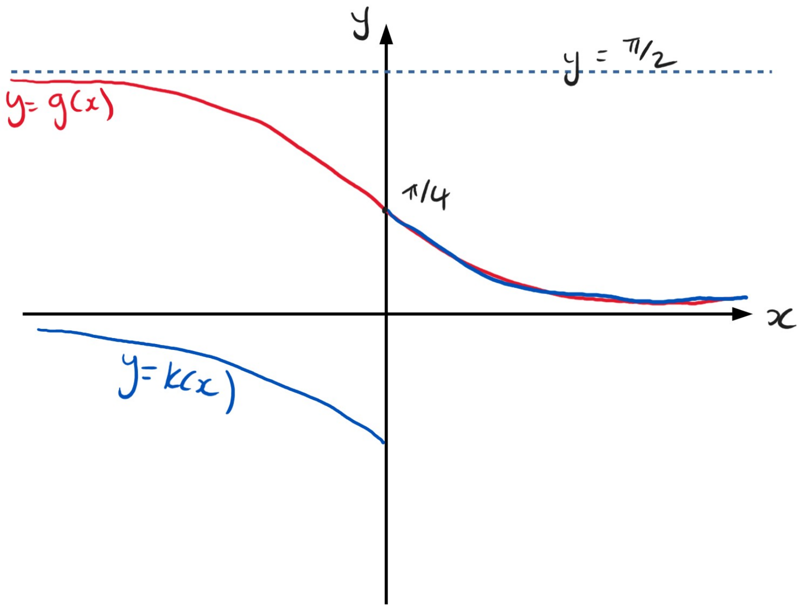
$$\begin{aligned}
\tan\big(\mathrm{k}(x)\big) &= \sqrt{x^2+1} - x \\
&= \mathrm{f}(x) \\
&= \tan\big(\tan^{-1}\big(\mathrm{f}(x)\big)\big) \\
&= \tan\big(\mathrm{g}(x)\big)
\end{aligned}$$

(d) For $x > 0$ we have $\mathrm{k}(x) = \mathrm{g}(x) = \tan^{-1}\left(\mathrm{f}(x)\right)$.

For $x < 0$ we can use parts **(ii)(a)** and **(ii)(b)** to get:

$$\mathrm{g}(-x) = \tfrac{1}{2}\pi - \mathrm{g}(x)$$
$$\mathrm{k}(-x) = -\mathrm{k}(x)$$

This gives us enough information to sketch the graphs:



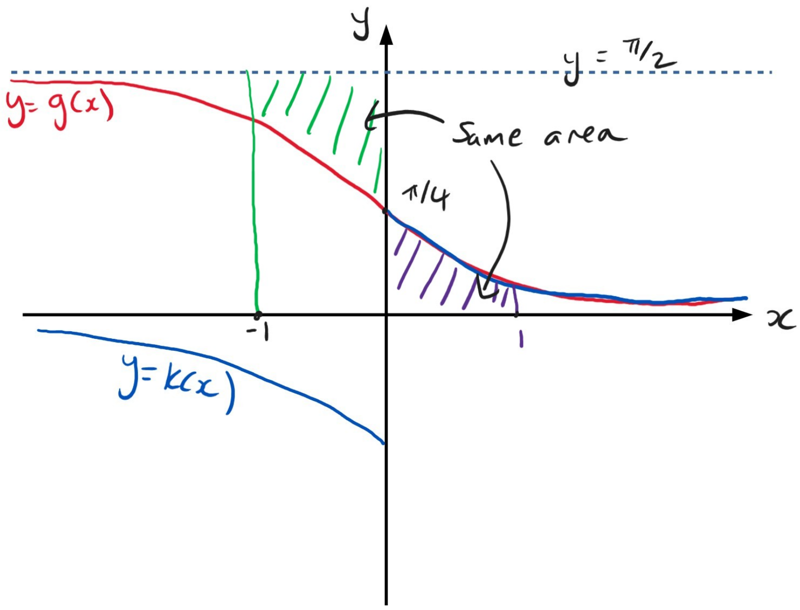
(e) We have:

$$\begin{aligned}\int_0^1 \mathrm{k}(x)\,\mathrm{d}x &= \int_0^1 \frac{1}{2}\tan^{-1}\left(\frac{1}{x}\right)\,\mathrm{d}x\\ &= \frac{1}{2}\left(\left[x\tan^{-1}\left(\frac{1}{x}\right)\right]_0^1 - \int_0^1 x \times \frac{1}{1+\left(\frac{1}{x}\right)^2} \times -x^{-2}\,\mathrm{d}x\right)\\ &= \frac{1}{2}\left(\frac{\pi}{4} + \int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x\right)\\ &= \frac{\pi}{8} + \frac{1}{2}\left[\frac{1}{2}\ln(1+x^2)\right]_0^1\\ &= \frac{\pi}{8} + \frac{\ln 2}{4}\end{aligned}$$

The derivative of $\tan^{-1} x$ can be found by using $\tan y = x$ and implicit differentiation, but it is a useful result to just "know" rather than deriving it each time.

Using our diagram from before we can see that:

$$\begin{aligned}\int_{-1}^0 \mathrm{g}(x)\,\mathrm{d}x &= \frac{\pi}{2} - \int_0^1 \mathrm{k}(x)\,\mathrm{d}x\\ &= \frac{\pi}{2} - \frac{\pi}{8} - \frac{\ln 2}{4}\\ &= \frac{3\pi}{8} - \frac{\ln 2}{4}\end{aligned}$$



The last command is “write down”, so no justification needed in obtaining the last integral!

Question 8

8 **(i)** Show that

$$z^{m+1} - \frac{1}{z^{m+1}} = \left(z - \frac{1}{z}\right)\left(z^m + \frac{1}{z^m}\right) + \left(z^{m-1} - \frac{1}{z^{m-1}}\right).$$

Hence prove by induction that, for $n \geqslant 1$,

$$z^{2n} - \frac{1}{z^{2n}} = \left(z - \frac{1}{z}\right)\sum_{r=1}^{n}\left(z^{2r-1} + \frac{1}{z^{2r-1}}\right).$$

Find similarly $z^{2n} - \dfrac{1}{z^{2n}}$ as a product of $\left(z + \dfrac{1}{z}\right)$ and a sum.

(ii) **(a)** By choosing $z = \mathrm{e}^{\mathrm{i}\theta}$, show that

$$\sin 2n\theta = 2\sin\theta \sum_{r=1}^{n} \cos(2r-1)\theta\,.$$

(b) Use this result, with $n = 2$, to show that $\cos\frac{2}{5}\pi = \cos\frac{1}{5}\pi - \frac{1}{2}$.

(c) Use this result, with $n = 7$, to show that $\cos\frac{2}{15}\pi + \cos\frac{4}{15}\pi + \cos\frac{8}{15}\pi + \cos\frac{16}{15}\pi = \frac{1}{2}$.

(iii) Show that $\sin\frac{1}{14}\pi - \sin\frac{3}{14}\pi + \sin\frac{5}{14}\pi = \frac{1}{2}$.

Examiner's report

In terms of the number of attempts, this was the second most popular question. The induction in part **(i)** was generally done very well with a clearly laid out proof. The fifth (method) mark in part **(i)** was often gained by finding a relevant identity. However, the final mark in part **(i)** was missed by a large majority of candidates due to not handling the alternating sign in the summation correctly. The summation is quite tricky in this respect, requiring the notation to be set up so that either the final term in the sum is positive, if terms are kept in the same order as for the previous part, or reversing the order of the sum (in which case the first term is positive).

Candidates often missed out on one or both marks in part **(ii)(a)** due to forgetting factors of 2 and i.

Most candidates that attempted parts **(ii)(b)** and **(ii)(c)** gained two method marks for successfully substituting a valid value of θ in both. However, a significant number of attempts at parts **(ii)(b)** and **(ii)(c)** did not gain full credit due to insufficient justification when manipulating trigonometric expressions. In general, candidates who stated the trigonometric identities they were using, or which specific terms were equivalent were successful here. Those that did multiple steps at once without justification often missed out on marks because it was not possible to pick out the results they had used. It is very significant here that the answer was given. Candidates should in general attempt to give more details when proving an answer given in the question to show they understand the intermediate steps between the starting point and given answer.

Most candidates did not attempt part **(iii)**. The successful attempts were from candidates who had given very clear answers to previous parts. There were a few different choices of n and θ that led to the required result.

Solution

(i) Starting on the right hand side we have:

$$\begin{aligned}&\left(z-\frac{1}{z}\right)\left(z^m+\frac{1}{z^m}\right)+\left(z^{m-1}-\frac{1}{z^{m-1}}\right)\\ =&z^{m+1}-z^{m-1}+\frac{1}{z^{m-1}}-\frac{1}{z^{m+1}}+\left(z^{m-1}-\frac{1}{z^{m-1}}\right)\\ =&z^{m+1}-\frac{1}{z^{m+1}}\end{aligned}$$

as required.

When $n=1$ we have:

$$\begin{aligned}&\left(z-\frac{1}{z}\right)\sum_{r=1}^{1}\left(z^{2r-1}+\frac{1}{z^{2r-1}}\right)\\ =&\left(z-\frac{1}{z}\right)\left(z^1+\frac{1}{z^1}\right)\\ =&z^2-\frac{1}{z^2}\end{aligned}$$

Therefore the statement is true when $n=1$.

Assume that the statement is true when $n=k$ so we have:

$$z^{2k}-\frac{1}{z^{2k}}=\left(z-\frac{1}{z}\right)\sum_{r=1}^{k}\left(z^{2r-1}+\frac{1}{z^{2r-1}}\right)$$

Now consider the case when $n=k+1$:

$$\begin{aligned}z^{2(k+1)}-\frac{1}{z^{2(k+1)}}&=z^{2k+2}-\frac{1}{z^{2k+2}}\\ &=\left(z-\frac{1}{z}\right)\left(z^{2k+1}+\frac{1}{z^{2k+1}}\right)+\left(z^{2k}-\frac{1}{z^{2k}}\right)\\ &=\left(z-\frac{1}{z}\right)\left(z^{2k+1}+\frac{1}{z^{2k+1}}\right)+\left(z-\frac{1}{z}\right)\sum_{r=1}^{k}\left(z^{2r-1}+\frac{1}{z^{2r-1}}\right)\\ &=\left(z-\frac{1}{z}\right)\sum_{r=1}^{k+1}\left(z^{2r-1}+\frac{1}{z^{2r-1}}\right)\end{aligned}$$

Which is the required result when $n = k+1$. Therefore if the statement is true when $n = k$ then it is true when $n = k+1$ and since it is true when $n = 1$ it is true for all integers $n \geqslant 1$.

For the second result first note that:

$$z^{m+1} - \frac{1}{z^{m+1}} = \left(z + \frac{1}{z}\right)\left(z^m - \frac{1}{z^m}\right) - \left(z^{m-1} - \frac{1}{z^{m-1}}\right)$$

Since this one is not a given result you don't have to show as much working, though it is a good idea to expand the right hand side in your head to double check the result!

CLAIM:

$$z^{2n} - \frac{1}{z^{2n}} = \left(z + \frac{1}{z}\right)\sum_{r=1}^{n}(-1)^{n-r}\left(z^{2r-1} - \frac{1}{z^{2r-1}}\right)$$

When $n = 1$ we have:

$$\begin{aligned} z^2 - \frac{1}{z^2} &= \left(z + \frac{1}{z}\right)\sum_{r=1}^{1}(-1)^{1-r}\left(z^{2r-1} - \frac{1}{z^{2r-1}}\right) \\ &= \left(z + \frac{1}{z}\right)\left(z^1 - \frac{1}{z^1}\right) \end{aligned}$$

Which is true.

Assume the statement is true for $n = k$ so we have:

$$z^{2k} - \frac{1}{z^{2k}} = \left(z + \frac{1}{z}\right)\sum_{r=1}^{k}(-1)^{n-r}\left(z^{2r-1} - \frac{1}{z^{2r-1}}\right)$$

Consider the $n = k+1$ case, so we have:

$$\begin{aligned} z^{2(k+1)} - \frac{1}{z^{2(k+1)}} &= z^{2k+2} - \frac{1}{z^{2k+2}} \\ &= \left(z + \frac{1}{z}\right)\left(z^{2k+1} - \frac{1}{z^{2k+1}}\right) - \left(z^{2k} - \frac{1}{z^{2k}}\right) \\ &= \left(z + \frac{1}{z}\right)\left(z^{2k+1} - \frac{1}{z^{2k+1}}\right) - \left(z + \frac{1}{z}\right)\sum_{r=1}^{k}(-1)^{n-r}\left(z^{2r-1} - \frac{1}{z^{2r-1}}\right) \\ &= \left(z + \frac{1}{z}\right)\left[\left(z^{2k+1} - \frac{1}{z^{2k+1}}\right) - \sum_{r=1}^{k}(-1)^{n-r}\left(z^{2r-1} - \frac{1}{z^{2r-1}}\right)\right] \\ &= \left(z + \frac{1}{z}\right)\sum_{r=1}^{k+1}(-1)^{n+1-r}\left(z^{2r-1} - \frac{1}{z^{2r-1}}\right) \end{aligned}$$

Therefore the claim is true.

(ii) **(a)** We know that $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ and $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$. Taking $z = e^{i\theta}$ we have:

$$\begin{aligned}
z^{2n} - \frac{1}{z^{2n}} &= e^{2ni\theta} - e^{-2ni\theta} \\
&= \left(e^{i\theta} - e^{i\theta}\right)\sum_{r=1}^{n}\left(e^{(2r-1)i\theta} + e^{-(2r-1)i\theta}\right) \\
\implies 2i\sin 2n\theta &= 2i\sin\theta\sum_{r=1}^{n} 2\cos(2r-1)\theta \\
\implies \sin 2n\theta &= 2\sin\theta\sum_{r=1}^{n}\cos(2r-1)\theta
\end{aligned}$$

(b) Taking $n = 2$ we have:

$$\begin{aligned}
\sin 4\theta &= 2\sin\theta\sum_{r=1}^{2}\cos(2r-1)\theta \\
\sin 4\theta &= 2\sin\theta\big(\cos\theta + \cos 3\theta\big)
\end{aligned}$$

Let $\theta = \dfrac{\pi}{5}$

$$\begin{aligned}
\sin\frac{4\pi}{5} &= 2\sin\frac{\pi}{5}\left(\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}\right) \\
\cancel{\sin\frac{\pi}{5}} &= 2\cancel{\sin\frac{\pi}{5}}\left(\cos\frac{\pi}{5} - \cos\frac{2\pi}{5}\right) \\
\implies \frac{1}{2} &= \cos\frac{\pi}{5} - \cos\frac{2\pi}{5}
\end{aligned}$$

Where the penultimate line above uses the facts that $\sin\theta = \sin(\pi - \theta)$ and $\cos\theta = -\cos(\pi-\theta)$. Therefore we have $\cos\frac{2\pi}{5} = \cos\frac{\pi}{5} - \frac{1}{2}$ as required.
These symmetries for sin and cos can just be stated without justification, but I tend to sketch out the graphs of sin and cos just to make sure I have the correct signs!

(c) Taking $n = 7$ we have:

$$\begin{aligned}
\sin 14\theta &= 2\sin\theta\sum_{r=1}^{7}\cos(2r-1)\theta \\
\sin 14\theta &= 2\sin\theta\big(\cos\theta + \cos 3\theta + \cdots + \cos 13\theta\big)
\end{aligned}$$

Let $\theta = \dfrac{\pi}{15}$:

$$\begin{aligned}
\sin\frac{14\pi}{15} &= 2\sin\frac{\pi}{15}\left(\cos\frac{\pi}{15} + \cos\frac{3\pi}{15} + \cdots + \cos\frac{13\pi}{15}\right) \\
\cancel{\sin\frac{\pi}{15}} &= 2\cancel{\sin\frac{\pi}{15}}\left(\cos\frac{\pi}{15} + \cos\frac{3\pi}{15} + \cdots + \cos\frac{13\pi}{15}\right)
\end{aligned}$$

At this stage a little pause for reflection is a good idea. We could replace some (or all) of the cos terms with their symmetric equivalents, but then it's quite hard to see how this would reduce the 7 cos terms to just 4.

We do have $\cos\frac{16\pi}{15} = \cos\left(\pi + \frac{\pi}{15}\right) = -\cos\frac{\pi}{15}$ which might be useful. We also know from the previous part[1] that $\cos\frac{2\pi}{5} = \cos\frac{\pi}{5} - \frac{1}{2}$ and so $\cos\frac{6\pi}{15} = \cos\frac{3\pi}{15} - \frac{1}{2}$. We also know that $\cos\frac{5\pi}{15} = \cos\frac{\pi}{3} = \frac{1}{2}$.

Using these results we have:

$$\begin{aligned}
\frac{1}{2} &= \cos\frac{\pi}{15} + \cos\frac{3\pi}{15} + \cos\frac{5\pi}{15} + \cos\frac{7\pi}{15} + \cos\frac{9\pi}{15} + \cos\frac{11\pi}{15} + \cos\frac{13\pi}{15} \\
\frac{1}{2} &= -\cos\frac{16\pi}{15} + \left(\cos\frac{6\pi}{15} + \frac{1}{2}\right) + \frac{1}{2} + \cos\frac{7\pi}{15} + \cos\frac{9\pi}{15} + \cos\frac{11\pi}{15} + \cos\frac{13\pi}{15} \\
-\frac{1}{2} &= -\cos\frac{16\pi}{15} + \cancel{\cos\frac{6\pi}{15}} - \cos\frac{8\pi}{15} - \cancel{\cos\frac{6\pi}{15}} - \cos\frac{4\pi}{15} - \cos\frac{2\pi}{15} \\
\implies \frac{1}{2} &= \cos\frac{16\pi}{15} + \cos\frac{8\pi}{15} + \cos\frac{4\pi}{15} + \cos\frac{2\pi}{15}
\end{aligned}$$

(iii) For this part we probably want to use the other identity we found in part **(i)**, since we haven't used it yet and this is the last part of the question!

Taking $z = e^{i\theta}$ we have:

$$\begin{aligned}
z^{2n} - \frac{1}{z^{2n}} &= \left(z + \frac{1}{z}\right)\sum_{r=1}^{n}(-1)^{n-r}\left(z^{2r-1} - \frac{1}{z^{2r-1}}\right) \\
\implies \sin(2n\theta) &= 2\cos\theta\sum_{r=1}^{n}(-1)^{n-r}\sin(2r-1)\theta
\end{aligned}$$

Taking $n = 3$ and $\theta = \frac{\pi}{14}$ gives:

$$\sin\frac{6\pi}{14} = 2\cos\frac{\pi}{14}\left(\sin\frac{\pi}{14} - \sin\frac{3\pi}{14} + \sin\frac{5\pi}{14}\right)$$

Since $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ this becomes:

$$\begin{aligned}
\cos\frac{\pi}{14} &= 2\cos\frac{\pi}{14}\left(\sin\frac{\pi}{14} - \sin\frac{3\pi}{14} + \sin\frac{5\pi}{14}\right) \\
\implies \frac{1}{2} &= \sin\frac{\pi}{14} - \sin\frac{3\pi}{14} + \sin\frac{5\pi}{14}
\end{aligned}$$

[1] One tip for getting unstuck on the later parts of a STEP question is to look back at the earlier parts and see if there is anything useful there that you can use!

Question 9

9 In this question, $n \geqslant 2$.

(i) A solid, of uniform density, is formed by rotating through $360°$ about the y-axis the region bounded by the part of the curve $r^{n-1}y = r^n - x^n$ with $0 \leqslant x \leqslant r$, and the x- and y-axes.

Show that the y-coordinate of the centre of mass of this solid is $\dfrac{nr}{2(n+1)}$.

(ii) Show that the normal to the curve $r^{n-1}y = r^n - x^n$ at the point $\big(rp, r(1-p^n)\big)$, where $0 < p \leqslant 1$, meets the y-axis at $(0, Y)$, where $Y = r\left(1 - p^n - \dfrac{1}{np^{n-2}}\right)$.

In the case $n = 4$, show that the greatest value of Y is $\frac{1}{4}r$.

(iii) A solid is formed by rotating through $360°$ about the y-axis the region bounded by the curves $r^3y = r^4 - x^4$ and $ry = -(r^2 - x^2)$, both for $0 \leqslant x \leqslant r$.

A and B are the points $(0, -r)$ and $(0, r)$, respectively, on the surface of the solid.

Show that the solid can rest in equilibrium on a horizontal surface with the vector $\overrightarrow{AB}$ at three different, non-zero, angles to the upward vertical. You should not attempt to find these angles.

Examiner's report

This question had the least number of attempts with a relatively small number of substantial attempts. Overall, this question was not done well, with only a small number of candidates achieving half marks or more. A significant number of candidates got 0, 1, or 2 marks in total for the question, with the main issue being getting started by knowing a suitable formula for the centre of mass.

In part **(i)**, many candidates overlooked fact that the curve was rotated about the y-axis, rather than the usual x-axis, and didn't change their formula to the correct variables.

Often candidates skipped straight to part **(ii)**, which was done generally done well. Part **(iii)** did not have many attempts. Of those candidates that did answer this part, most were only able to get the first few marks for finding the centre of mass of the full shape. The final part (showing there are multiple ways to balance the solid) was not attempted enough to observe any patterns but it was clear that candidates struggled to demonstrate clear understanding of how the equilibrium condition relates to the position of the centre of mass.

Solution

(i) Using the formula for the y coordinate of the centre of mass we have:

$$\bar{y} = \frac{\int \pi y x^2 \, \mathrm{d}y}{\int \pi x^2 \, \mathrm{d}y} = \frac{\int y x^2 \, \mathrm{d}y}{\int x^2 \, \mathrm{d}y}$$

Rearranging the equation of the curve gives:

$$\begin{aligned} x^n &= r^n - r^{n-1}y \\ x^n &= r^n \left(1 - \frac{y}{r}\right) \\ x &= r\left(1 - \frac{y}{r}\right)^{\frac{1}{n}} \end{aligned}$$

Working out the integrals:

$$\begin{aligned} \int_0^r x^2 \, \mathrm{d}y &= \int_0^r r^2 \left(1 - \frac{y}{r}\right)^{\frac{2}{n}} \, \mathrm{d}y \\ &= \left[\frac{-r^3}{\left(\frac{2}{n}+1\right)}\left(1 - \frac{y}{r}\right)^{\frac{2}{n}+1}\right]_0^r \\ &= \frac{r^3}{\left(\frac{2}{n}+1\right)} \end{aligned}$$

and:

$$\begin{aligned} \int_0^r y x^2 \, \mathrm{d}y &= r^2 \int_0^r y \left(1 - \frac{y}{r}\right)^{\frac{2}{n}} \, \mathrm{d}y \\ &= r^2 \left[\frac{-ry}{\left(\frac{2}{n}+1\right)}\left(1 - \frac{y}{r}\right)^{\frac{2}{n}+1}\right]_0^r + \frac{r^3}{\left(\frac{2}{n}+1\right)} \int_0^r \left(1 - \frac{y}{r}\right)^{\frac{2}{n}+1} \mathrm{d}y \\ &= \frac{r^3}{\left(\frac{2}{n}+1\right)} \left[\frac{-r}{\left(\frac{2}{n}+2\right)}\left(1 - \frac{y}{r}\right)^{\frac{2}{n}+2}\right]_0^r \\ &= \frac{r^4}{\left(\frac{2}{n}+1\right)\left(\frac{2}{n}+2\right)} \end{aligned}$$

Therefore we have:

$$\begin{aligned} \bar{y} &= \frac{\int y x^2 \, \mathrm{d}y}{\int x^2 \, \mathrm{d}y} \\ &= \frac{r^4}{\cancel{\left(\frac{2}{n}+1\right)}\left(\frac{2}{n}+2\right)} \times \frac{\cancel{\left(\frac{2}{n}+1\right)}}{r^3} \\ &= \frac{r}{\frac{2}{n}+2} \\ &= \frac{nr}{2+2n} \\ &= \frac{nr}{2(n+1)} \end{aligned}$$

as required.

(ii) We have $y = r - \frac{x^n}{r^{n-1}}$. Differentiation gives $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{nx^{n-1}}{r^{n-1}}$.

The equation of the normal through the point with $x = rp$ and $y = r(1-p^n)$ is:

$$\begin{aligned} y - r(1-p^n) &= \frac{r^{n-1}}{n(rp)^{n-1}}(x - rp) \\ y - r(1-p^n) &= \frac{1}{np^{n-1}}(x - rp) \\ x = 0 \implies Y &= r(1-p^n) - \frac{rp}{np^{n-1}} \\ Y &= r\left(1 - p^n - \frac{1}{np^{n-2}}\right) \end{aligned}$$

When $n = 4$ we have:

$$Y = r\left(1 - p^4 - \frac{1}{4p^2}\right)$$

Differentiating gives:

$$\begin{aligned} \frac{\mathrm{d}Y}{\mathrm{d}p} &= r\left(-4p^3 + \frac{1}{2p^3}\right) \\ 0 &= r\left(\frac{1}{2p^3} - 4p^3\right) \\ \implies p &= \left(\tfrac{1}{8}\right)^{\frac{1}{6}} = \left(\tfrac{1}{2}\right)^{\frac{1}{2}} \end{aligned}$$

Since the second derivative is $\frac{\mathrm{d}^2Y}{\mathrm{d}p^2} = r\left(-12p^2 - \frac{3}{2p^4}\right) < 0$ then this value of p will maximise Y.

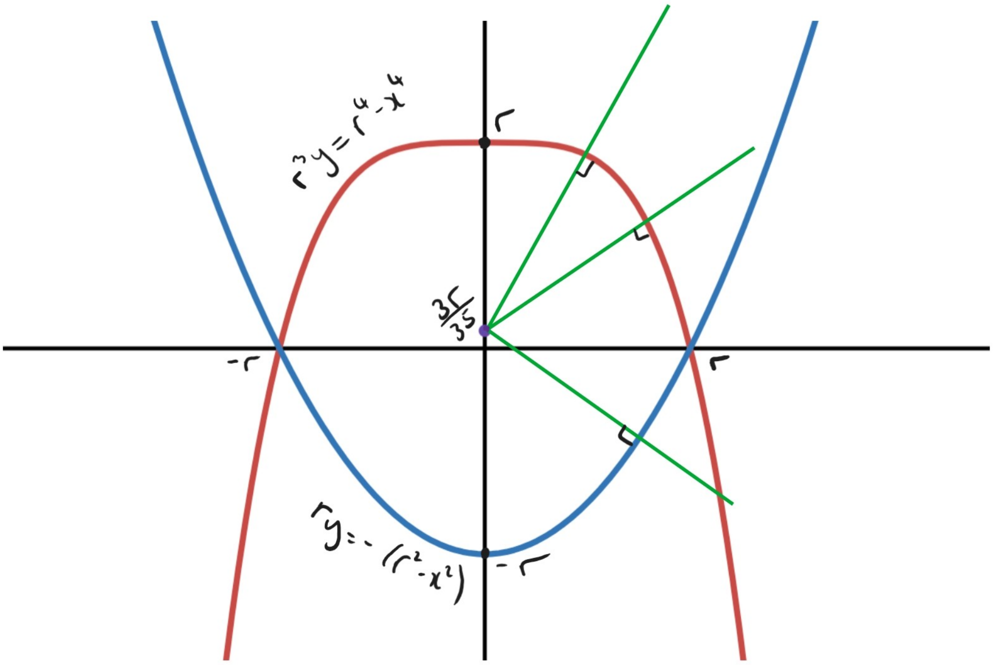
Therefore:

$$\begin{aligned} Y_{\max} &= r\left(1 - p^4 - \frac{1}{4p^2}\right) \\ &= r\left(1 - \frac{1}{4} - \frac{1}{4 \times \frac{1}{2}}\right) \\ &= r\left(1 - \frac{1}{4} - \frac{1}{2}\right) \\ &= \frac{r}{4} \end{aligned}$$

(iii) A diagram is not strictly necessary for this question, but might help to get an understanding of the situation.

The last paragraph is rather wordy, but is basically saying that there are three different positions where the shape can balance and not tip over. The wording of the paragraph has been chosen to make this idea mathematically precise.

If a point on the surface is a "balancing point" (i.e. the solid can rest in equilibrium at that point) then the normal to the surface at that point must pass through the centre of mass of the whole solid.



Note that the points $(0, r)$ and $(0, -r)$ are "balancing points", but these are excluded by the requirement for the axis of symmetry of the solid to make a non-zero angle to the upwards vertical.

From part **(i)** we have the y coordinate of the centre of mass equal to $\bar{y} = \dfrac{nr}{2(n+1)}$. Therefore the centre of mass of the "top" part of the solid is at $\left(0, \frac{4r}{2(4+1)}\right) = \left(0, \frac{2}{5}r\right)$ and the centre of mass of the "bottom" part is at $\left(0, -\frac{2r}{2(2+1)}\right) = \left(0, -\frac{1}{3}r\right)$.

If the mass of the top part is m_1 and the mass of the bottom part is m_2 then the y coordinate of the centre of mass of the whole solid is given by:

$$\bar{y} = \frac{\frac{2}{5}r \times m_1 - \frac{1}{3}r \times m_2}{m_1 + m_2}$$

We know that the mass of a shape is proportional to the volume, which in turn is proportional to $\int x^2 \mathrm{d}y$. Therefore we have:

$$\begin{aligned} m_1 &= k \int x^2 \mathrm{d}y \\ &= \frac{kr^3}{\left(\frac{2}{4} + 1\right)} \\ m_2 &= \frac{kr^3}{\left(\frac{2}{2} + 1\right)} \end{aligned}$$

Where here we are using the expression for $\int x^2 \, \mathrm{d}y = \dfrac{r^3}{\left(\frac{2}{n} + 1\right)}$ found in part **(i)**.

So we have:

$$\begin{aligned}
\bar{y} &= \frac{\frac{2}{5}r \times m_1 - \frac{1}{3}r \times m_2}{m_1 + m_2} \\
&= \frac{\frac{2}{5}r \times \frac{kr^3}{\left(\frac{2}{4}+1\right)} - \frac{1}{3}r \times \frac{kr^3}{\left(\frac{2}{2}+1\right)}}{\frac{kr^3}{\left(\frac{2}{4}+1\right)} + \frac{kr^3}{\left(\frac{2}{2}+1\right)}} \\
&= \frac{\frac{2}{5}r \times \frac{2}{3} - \frac{1}{3}r \times \frac{1}{2}}{\frac{2}{3} + \frac{1}{2}} \\
&= \frac{8r - 5r}{20 + 15} \\
&= \frac{3r}{35}
\end{aligned}$$

For a point on the edge of the solid to be a “balancing point” we need to normal to that point to pass through $\left(0, \frac{3r}{35}\right)$. Hence we need to find the number of solutions to $Y = \frac{3r}{35}$, and we want solutions for p that satisfy $0 < p \leqslant 1$. Note that we don’t need to actually find the solutions.

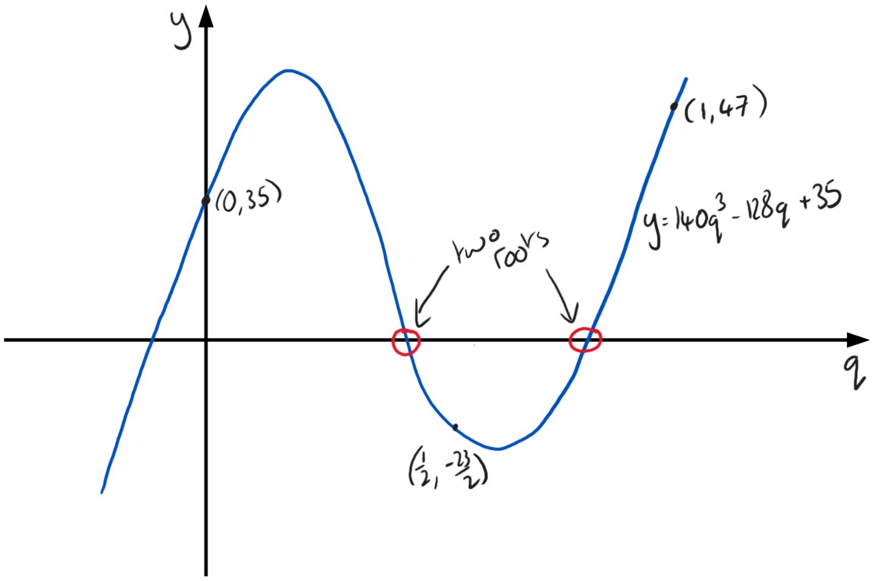
For the “top” part this means we need:

$$\begin{aligned}
\cancel{r}\left(1 - p^4 - \frac{1}{4p^2}\right) &= \frac{3\cancel{r}}{35} \\
140p^2 - 140p^6 - 35 &= 12p^2 \\
140p^6 - 128p^2 + 35 &= 0
\end{aligned}$$

If we let $p^2 = q$ this becomes a cubic equation $140q^3 - 128q + 35 = 0$.

We know that this cubic $y = 140q^3 - 128q + 35$ passes through $(0, 35)$ and if you take $q = 1$ it passes through $(1, 47)$.

If we try $q = \frac{1}{2}$ we find that the curve passes through $\left(\frac{1}{2}, -\frac{23}{2}\right)$, and since it is a continuous curve there must be two roots, one with $q \in \left(0, \frac{1}{2}\right)$ and one with $q \in \left(\frac{1}{2}, 1\right)$. Therefore there are two possible values of p satisfying $0 < p \leqslant 1$.



For the "bottom" part of the curve we need:

$$
\begin{aligned}
-r\left(1 - p^2 - \frac{1}{2}\right) &= \frac{3r}{35} \\
-(70 - 70p^2 - 35) &= 6 \\
70p^2 &= 41 \\
p &= \sqrt{\frac{41}{70}}
\end{aligned}
$$

Therefore this is a third "balancing" point, and so we have three balancing points in total.

Question 10

10 A plank AB of length L initially lies horizontally at rest along the x-axis on a flat surface, with A at the origin.

Point C on the plank is such that AC has length sL, where $0 < s < 1$.

End A is then raised vertically along the y-axis so that its height above the horizontal surface at time t is $\mathrm{h}(t)$, while end B remains in contact with the flat surface and on the x-axis.

The function $\mathrm{h}(t)$ satisfies the differential equation

$$\frac{\mathrm{d}^2\mathrm{h}}{\mathrm{d}t^2} = -\omega^2\mathrm{h}, \quad \text{with } \mathrm{h}(0) = 0 \text{ and } \frac{\mathrm{dh}}{\mathrm{d}t} = \omega L \text{ at } t = 0,$$

where ω is a positive constant.

A particle P of mass m remains in contact with the plank at point C.

(i) Show that the x-coordinate of P is $sL\cos\omega t$, and find a similar expression for its y-coordinate.

(ii) Find expressions for the x- and y-components of the acceleration of the particle.

(iii) N and F are the upward normal and frictional components, respectively, of the force of the plank on the particle. Show that

$$N = mg(1 - k\sin\omega t)\cos\omega t,$$

and that

$$F = mgsk + N\tan\omega t$$

where $k = \dfrac{L\omega^2}{g}$.

(iv) The coefficient of friction between the particle and the plank is $\tan\alpha$, where α is an acute angle.

Show that the particle will not slip initially, provided $sk < \tan\alpha$.

Show further that, in this case, the particle will slip

- while N is still positive,
- when the plank makes an angle less than α to the horizontal.

Examiner's report

This question did not receive very many attempts.

Part **(i)** was generally done well, although a significant portion of candidates did not draw a diagram and didn't correctly calculate y, i.e. not performing the necessary subtraction. This meant errors followed through to subsequent parts.

Part **(ii)** was generally done well.

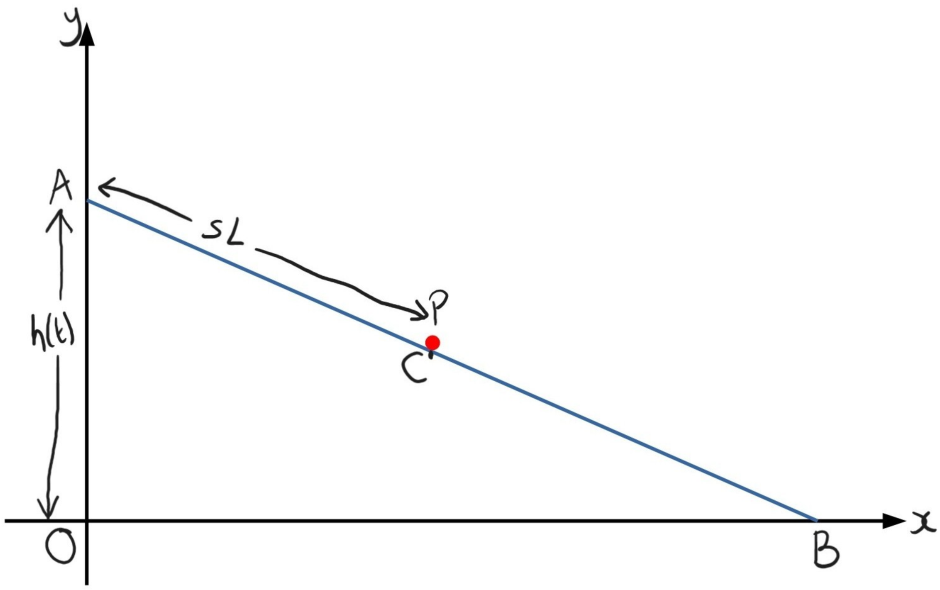
A good portion of those that attempted part **(iii)** realised they had to resolve in two directions (either horizontal and vertical or parallel and perpendicular), and made a good attempt to do this. Choosing to resolve horizontally and vertically proved to be more straightforward. Those that chose to resolve parallel and perpendicular often had some difficulties with calculating the resultant acceleration.

Part **(iv)** was not answered successfully on the whole. Considering the equivalent problem when the plank is not moving may have led to considering $t = \frac{\alpha}{\omega}$, leading to the key idea that $F < \mu N$ when $t = 0$ and $F > \mu N$ when $t = \frac{\alpha}{\omega}$.

Solution

The equations for $\mathrm{h}(t)$ are SHM equations, but we are only considering the first "up and down" of A, as A cannot pass below the surface.

The setup looks like this:

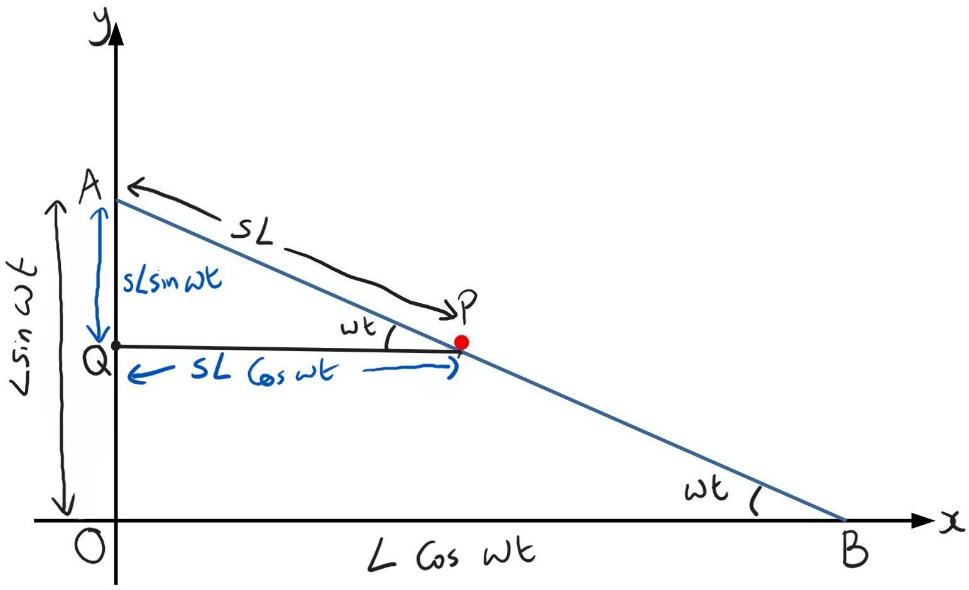


As in almost all mechanics questions a good starting point is drawing a (clear!) diagram. Rules are to be encouraged.

(i) The general solution of the SHM equation $\frac{\mathrm{d}^2\mathrm{h}}{\mathrm{d}t^2} = -\omega^2\mathrm{h}$ is $\mathrm{h}(t) = P\cos\omega t + Q\sin\omega t$ (or you could use $\mathrm{h}(t) = R\sin(\omega t + \alpha)$, or $\mathrm{h}(t) = R\cos(\omega t + \alpha)$).

Using $\mathrm{h}(0) = 0$ gives $P = 0$, and using $\frac{\mathrm{dh}}{\mathrm{d}t} = \omega L$ when $t = 0$ gives $Q = L$. Therefore we have $\mathrm{h}(t) = L\sin\omega t$. This means that at time t we have $\angle OBA = \omega t$.

Adding on a horizontal line at the point where P is on the slope gives two similar triangles.



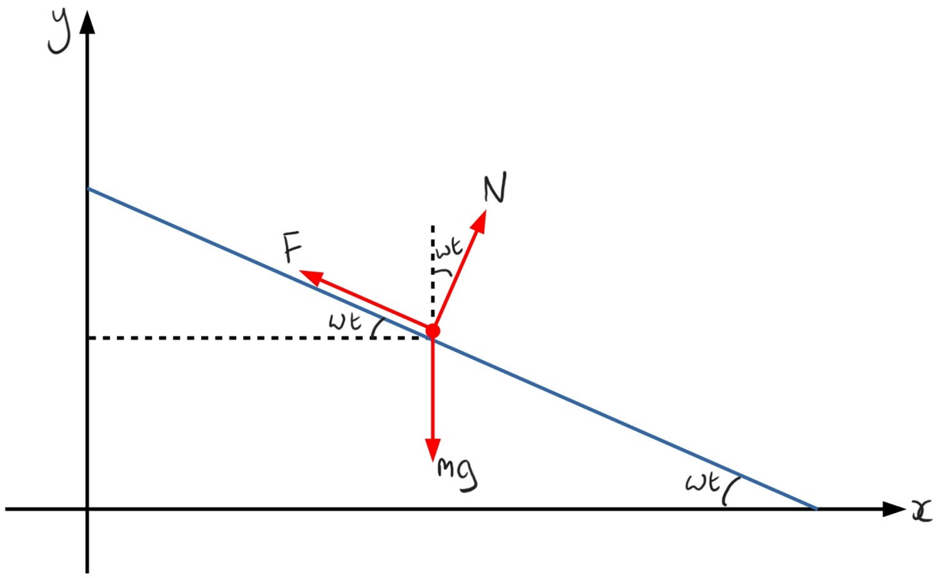
The lengths on the smaller QPA triangle are all multiples of the lengths of the larger OBA trangle, with a scale factor of s.

The coordinates of P are $\big(sL\cos(\omega t), L\sin(\omega t) - Ls\sin(\omega t)\big) = \big(sL\cos(\omega t), (1-s)L\sin(\omega t)\big)$.

(ii) Differentiating with respect to time twice gives:

$$\ddot{x} = -\omega^2 sL\cos(\omega t)$$
$$\ddot{y} = -\omega^2(1-s)L\sin(\omega t)$$

(iii) The forces acting on the particle are shown below:



Using $F = ma$ horizontally we have:

$$m\big(-sL\omega^2\cos(\omega t)\big) = N\sin(\omega t) - F\cos(\omega t)$$
$$\implies F\cos(\omega t) = N\sin(\omega t) + msL\omega^2\cos(\omega t)$$

And using $F = ma$ vertically we have:

$$m\big(-(1-s)L\omega^2\sin(\omega t)\big) = N\cos(\omega t) + F\sin(\omega t) - mg$$

Substituting for F in this last equation gives:

$$m\big(-(1-s)L\omega^2\sin(\omega t)\big) = N\cos(\omega t) + \frac{\sin(\omega t)}{\cos(\omega t)}\Big[N\sin(\omega t) + msL\omega^2\cos(\omega t)\Big] - mg$$
$$m(s-1)L\omega^2\sin(\omega t)\cos(\omega t) = N\cos^2(\omega t) + \sin(\omega t)\Big[N\sin(\omega t) + msL\omega^2\cos(\omega t)\Big] - mg\cos(\omega t)$$
$$-mL\omega^2\sin(\omega t)\cos(\omega t) = N\cos^2(\omega t) + N\sin^2(\omega t) - mg\cos(\omega t)$$
$$\implies N = mg\cos(\omega t) - mL\omega^2\sin(\omega t)\cos(\omega t)$$
$$N = mg\cos(\omega t)\left(1 - \frac{L\omega^2}{g}\sin(\omega t)\right)$$
$$N = mg\cos(\omega t)\big(1 - k\sin(\omega t)\big)$$

as required.

Using the horizontal expression for F:

$$F\cos(\omega t) = N\sin(\omega t) + msL\omega^2\cos(\omega t)$$
$$F = N\tan(\omega t) + msL\omega^2$$
$$F = N\tan(\omega t) + msg\frac{L\omega^2}{g}$$
$$F = N\tan(\omega t) + msgk$$

as required.

(iv) Initially $t = 0$ and then we have $N = mg$ and $F = mgsk$.

We have:

$$\begin{aligned} F - \mu N &= F - \mu mg \\ &= mgsk - \mu mg \\ &= mg(sk - \mu) \end{aligned}$$

and as long as this is negative then F is less than the limiting friction and the particle will not slip. Hence it won't slip as long as $sk < \mu \implies sk < \tan\alpha$.

Now consider what happens when $t = \frac{\alpha}{\omega}$. At this time we have:

$$F = mgsk + N\tan\alpha$$

This gives $F - \mu N = mgsk > 0$, and so $F > \mu N$ and the particle has slipped before $t = \frac{\alpha}{\omega}$, i.e. we have $\omega t < \alpha$ and so the particle slips before the plank makes an angle of α to the horizontal.

Therefore there exists a time T with $0 < T < \frac{\alpha}{\omega}$ with $F = \mu N$ at which point the particle slips. At this time we have:

$$\begin{aligned} \frac{F - \mu N}{mg} &= \frac{mgsk + N\tan\omega T - \mu N}{mg} \\ 0 &= sk + \frac{N}{mg}(\tan\omega T - \tan\alpha) \\ \implies \frac{N}{mg} &= \frac{sk}{\tan\alpha - \tan\omega T} \end{aligned}$$

We know that m, g, s, and k are all positive, and since $0 < T < \frac{\alpha}{\omega} \implies 0 < \omega T < \alpha < 90°$ (since we know that α is acute) we know that $\tan\alpha > \tan\omega T$.

Therefore we know that we have $N > 0$.

Question 11

11 **(i)** Let $\lambda > 0$. The independent random variables $X_1, X_2, \ldots, X_n$ all have probability density function

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geqslant 0 \\ 0 & t < 0 \end{cases}$$

and cumulative distribution function $F(x)$.

The value of random variable Y is the largest of the values $X_1, X_2, \ldots, X_n$. Show that the cumulative distribution function of Y is given, for $y \geqslant 0$, by

$$G(y) = \left(1 - e^{-\lambda y}\right)^n .$$

(ii) The values $L(\alpha)$ and $U(\alpha)$, where $0 < \alpha \leqslant \frac{1}{2}$, are such that

$$P(Y < L(\alpha)) = \alpha \quad \text{and} \quad P(Y > U(\alpha)) = \alpha.$$

Show that

$$L(\alpha) = -\frac{1}{\lambda} \ln \left(1 - \alpha^{\frac{1}{n}}\right)$$

and write down a similar expression for $U(\alpha)$.

(iii) Use the approximation $e^t \approx 1 + t$, for $|t|$ small, to show that, for sufficiently large n,

$$\lambda L(\alpha) \approx \ln(n) - \ln\left(\ln\left(\frac{1}{\alpha}\right)\right).$$

(iv) Hence show that the median of Y tends to infinity as n increases, but that the width of the interval $U(\alpha) - L(\alpha)$ tends to a value which is independent of n.

(v) You are given that, for $|t|$ small, $\ln(1 + t) \approx t$ and that $e^3 \approx 20$.

Show that, for sufficiently large n, there is an interval of width approximately $4\lambda^{-1}$ in which Y lies with probability 0.9.

Examiner's report

Parts **(i)** and **(ii)** were generally well done.

Candidates were often able to make good progress with parts **(iv)** and **(v)** even if they had found difficulty with part **(iii)** (since the answer to part **(iii)** was given in the question).

In part **(iii)**, many candidates incorrectly assumed that $\alpha^{\frac{1}{n}} \to 0$, leading to the incorrect approximation $\ln\left(1 - \alpha^{\frac{1}{n}}\right) \approx -\alpha^{\frac{1}{n}}$.

A significant number of candidates ignored the word 'hence' in part **(iv)**, either:

- not realising that $\mathrm{L}\left(\frac{1}{2}\right)$ was the median, instead solving $\mathrm{G}(m) = \frac{1}{2}$ directly, or
- Writing down the exact expression for $\mathrm{U}(\alpha) - \mathrm{L}(\alpha)$ and attempting to find the limit of this as $n \to \infty$.

Most candidates who attempted part **(v)** focused entirely on estimating the size of $\mathrm{U}(0.05) - \mathrm{L}(0.05)$, without ever stating that $\mathrm{P}\big(\mathrm{L}(0.05) < Y < \mathrm{U}(0.05)\big) = 0.9$.

In parts **(iii)**, **(iv)** and **(v)**, a number of candidates did not give sufficient precision in the use of approximations/limits, for example writing asymptotic results as equalities which held for all n.

Approximately half of the candidates implicitly utilised the identity $\mathrm{U}(\alpha) = \mathrm{L}(1 - \alpha)$. Whilst, formally, the bound $0 < \alpha < \frac{1}{2}$ given in the question invalidated this method unless the range of the arguments of L and U were first extended to $0 < \alpha < 1$, the identity allowed candidates to save considerable repetition of work and candidates who employed this method were not penalised on account of this technical subtlety.

Solution

(i) We have:

$$\begin{aligned} \mathrm{P}(X_1 \leqslant t) &= \int_0^t \lambda \mathrm{e}^{-\lambda x}\, \mathrm{d}x \\ &= \Big[-\mathrm{e}^{-\lambda x} \Big]_0^t \\ &= 1 - \mathrm{e}^{-\lambda t} \end{aligned}$$

Since Y is the largest of the X_i and all the X_i are independent then we have:

$$\begin{aligned} \mathrm{P}(Y \leqslant y) &= \mathrm{P}(X_1 \leqslant y) \times \mathrm{P}(X_2 \leqslant y) \times \cdots \times \mathrm{P}(X_n \leqslant y) \\ &= \left(1 - \mathrm{e}^{-\lambda y}\right)^n \end{aligned}$$

(ii) The notation perhaps looks a little odd, but $L(\alpha)$ is a "Lower" value such that the probability that Y is less than $L(\alpha)$ is equal to α. Similarly $U(\alpha)$ is the "Upper" value such that the probability that Y is greater than $U(\alpha)$ is also equal to α.

We have:

$$\begin{aligned} \mathrm{P}\big(Y < \mathrm{L}(\alpha)\big) &= \alpha \\ \left(1 - \mathrm{e}^{-\lambda \mathrm{L}(\alpha)}\right)^n &= \alpha \\ 1 - \mathrm{e}^{-\lambda \mathrm{L}(\alpha)} &= \alpha^{\frac{1}{n}} \\ \mathrm{e}^{-\lambda \mathrm{L}(\alpha)} &= 1 - \alpha^{\frac{1}{n}} \\ \mathrm{L}(\alpha) &= -\frac{1}{\lambda} \ln\left(1 - \alpha^{\frac{1}{n}}\right) \end{aligned}$$

and

$$\begin{aligned} \mathrm{P}\big(Y > \mathrm{U}(\alpha)\big) &= \alpha \\ 1 - \left(1 - \mathrm{e}^{-\lambda \mathrm{U}(\alpha)}\right)^n &= \alpha \\ 1 - \mathrm{e}^{-\lambda \mathrm{U}(\alpha)} &= (1-\alpha)^{\frac{1}{n}} \\ \mathrm{e}^{-\lambda \mathrm{U}(\alpha)} &= 1 - (1-\alpha)^{\frac{1}{n}} \\ \mathrm{U}(\alpha) &= -\frac{1}{\lambda} \ln\left(1 - (1-\alpha)^{\frac{1}{n}}\right) \end{aligned}$$

(iii) Note that as $n \to \infty$, $\alpha^{\frac{1}{n}} \to 1$, and so $\left|1 - \alpha^{\frac{1}{n}}\right|$ is small. We have:

$$\begin{aligned} \mathrm{e}^{-\lambda \mathrm{L}(\alpha)} &= \mathrm{e}^{\ln\left(1-\alpha^{\frac{1}{n}}\right)} \\ \mathrm{e}^{-\lambda \mathrm{L}(\alpha)} &= \ln \mathrm{e}^{\left(1-\alpha^{\frac{1}{n}}\right)} \\ \mathrm{e}^{-\lambda \mathrm{L}(\alpha)} &= \ln\left[\frac{1}{\mathrm{e}^{\left(\alpha^{\frac{1}{n}}-1\right)}}\right] \\ \mathrm{e}^{-\lambda \mathrm{L}(\alpha)} &\approx \ln\left[\frac{1}{1 + \left(\alpha^{\frac{1}{n}} - 1\right)}\right] \\ \mathrm{e}^{-\lambda \mathrm{L}(\alpha)} &\approx \ln\left[\frac{1}{\alpha^{\frac{1}{n}}}\right] \\ \mathrm{e}^{-\lambda \mathrm{L}(\alpha)} &\approx \ln\left[\alpha^{-\frac{1}{n}}\right] \\ \mathrm{e}^{-\lambda \mathrm{L}(\alpha)} &\approx \frac{-1}{n} \ln(\alpha) \\ n\mathrm{e}^{-\lambda \mathrm{L}(\alpha)} &\approx \ln(\alpha^{-1}) \\ \ln n - \lambda \mathrm{L}(\alpha) &\approx \ln\left(\ln(\alpha^{-1})\right) \\ \implies \lambda \mathrm{L}(\alpha) &\approx \ln n - \ln\left[\ln\left(\frac{1}{\alpha}\right)\right] \end{aligned}$$

The second line follows since e^x and $\ln x$ are inverses.

This is the first way I solved this part, but there are quicker ways!

Alternative method:

We have:

$$\begin{aligned}\alpha^{\frac{1}{n}} &= \mathrm{e}^{\ln\left(\alpha^{\frac{1}{n}}\right)}\\ &= \mathrm{e}^{\frac{\ln\alpha}{n}}\\ &\approx 1 + \frac{\ln\alpha}{n} \quad \text{since } \frac{\ln\alpha}{n} \text{ is small}\end{aligned}$$

Using this we have:

$$\begin{aligned}\lambda\mathrm{L}(\alpha) &= -\ln\left(1-\alpha^{\frac{1}{n}}\right)\\ \lambda\mathrm{L}(\alpha) &\approx -\ln\left[1-\left(1+\frac{\ln\alpha}{n}\right)\right]\\ \lambda\mathrm{L}(\alpha) &\approx -\ln\left[-\frac{\ln\alpha}{n}\right]\\ \lambda\mathrm{L}(\alpha) &\approx -\ln\left[\frac{-\ln\alpha}{n}\right]\\ \lambda\mathrm{L}(\alpha) &\approx -\Big[\ln(-\ln\alpha) - (\ln n)\Big]\\ \lambda\mathrm{L}(\alpha) &\approx \ln n - \ln(\ln\alpha^{-1})\end{aligned}$$

(iv) The median of Y is is the value m such that $\mathrm{P}(Y < m) = \mathrm{P}(Y > m) = \frac{1}{2}$. This means that it is equal to $L\left(\frac{1}{2}\right)$, or equivalently $U\left(\frac{1}{2}\right)$.

When n is large we have:

$$\mathrm{L}\left(\tfrac{1}{2}\right) \approx \frac{1}{\lambda}\big(\ln n - \ln(\ln 2)\big)$$

and so $\mathrm{L}\left(\frac{1}{2}\right) \to \infty$ as $n \to \infty$, i.e. the median of Y tends to infinity as n increases.

Similarly to before:

$$\begin{aligned}(1-\alpha)^{\frac{1}{n}} &= \mathrm{e}^{\ln\left((1-\alpha)^{\frac{1}{n}}\right)}\\ &= \mathrm{e}^{\frac{\ln(1-\alpha)}{n}}\\ &\approx 1 + \frac{\ln(1-\alpha)}{n}\end{aligned}$$

Using this we have:

$$\begin{aligned}\lambda\mathrm{U}(\alpha) &= -\ln\left(1-(1-\alpha)^{\frac{1}{n}}\right)\\ \lambda\mathrm{U}(\alpha) &\approx -\ln\left[1-\left(1+\frac{\ln(1-\alpha)}{n}\right)\right]\\ \lambda\mathrm{U}(\alpha) &\approx -\ln\left[-\frac{\ln(1-\alpha)}{n}\right]\\ \lambda\mathrm{U}(\alpha) &\approx -\ln(-\ln(1-\alpha)) - (-\ln n)\\ \lambda\mathrm{U}(\alpha) &\approx \ln n - \ln(\ln(1-\alpha)^{-1})\end{aligned}$$

Therefore when n is large we have:

$$\lambda\big(\mathrm{U}(\alpha) - \mathrm{L}(\alpha)\big) \approx \Big[\ln n - \ln(\ln(1-\alpha)^{-1})\Big] - \Big[\ln n - \ln(\ln \alpha^{-1})\Big]$$

$$\implies \mathrm{U}(\alpha) - \mathrm{L}(\alpha) \approx \frac{1}{\lambda}\Big[\ln(\ln \alpha^{-1}) - \ln\big(\ln(1-\alpha)^{-1}\big)\Big]$$

which is independent of n.

(v) We know that $\mathrm{P}\big(Y < \mathrm{L}(0.05)\big) = 0.05$ and $\mathrm{P}\big(Y > \mathrm{U}(0.05)\big) = 0.05$.
Therefore $\mathrm{P}\big(\mathrm{L}(0.05) < Y < \mathrm{U}(0.05)\big) = 0.9$.

The width of the interval (using the expression found at the end of part **(iv)**) is approximately:

$$\mathrm{U}(0.05) - \mathrm{L}(0.05) \approx \frac{1}{\lambda}\left[\ln(\ln 20) - \ln\left(\ln\frac{20}{19}\right)\right]$$

$$\mathrm{U}(0.05) - \mathrm{L}(0.05) \approx \frac{1}{\lambda}\left[\ln(\ln 20) - \ln\left(\ln\left(1 + \frac{1}{19}\right)\right)\right]$$

$$\mathrm{U}(0.05) - \mathrm{L}(0.05) \approx \frac{1}{\lambda}\left[\ln(3) - \ln\left(\frac{1}{19}\right)\right]$$

Where the last lie uses $\mathrm{e}^3 \approx 20 \implies \ln 20 \approx 3$ and $\ln(1+t) \approx t$ (assuming that $\frac{1}{19}$ is sufficiently "small").

Continuing the approximations:

$$\mathrm{U}(0.05) - \mathrm{L}(0.05) \approx \frac{1}{\lambda}\left[\ln 3 - (\ln 1 - \ln 19)\right]$$

$$\mathrm{U}(0.05) - \mathrm{L}(0.05) \approx \frac{1}{\lambda}\left[\ln 3 + \ln 19\right]$$

$$\mathrm{U}(0.05) - \mathrm{L}(0.05) \approx \frac{1}{\lambda}\left[\ln 57\right]$$

Now all we need is to show that $\ln 57 \approx 4$. We know that $\ln 20 \approx 3$, so let's try to use that:

$$\ln 57 = \ln\left(20 \times \frac{57}{20}\right)$$

$$\ln 57 = \ln 20 + \ln\left(\frac{57}{20}\right)$$

$$\ln 57 \approx 3 + \ln 2.85$$

and as $2.85 \approx \mathrm{e}$ we have $\ln 57 \approx 4$ as required.

Question 12

12 **(i)** Show that, for any functions f and g, and for any $m \geqslant 0$,

$$\sum_{r=1}^{m+1}\left(\mathrm{f}(r)\sum_{s=r-1}^{m}\mathrm{g}(s)\right)=\sum_{s=0}^{m}\left(\mathrm{g}(s)\sum_{r=1}^{s+1}\mathrm{f}(r)\right).$$

(ii) The random variables $X_0, X_1, X_2, \ldots$ are defined as follows

- X_0 takes the value 0 with probability 1;
- X_{n+1} takes the values $0, 1, \ldots, X_n + 1$ with equal probability, for $n = 0, 1, \ldots$.

(a) Write down $\mathrm{E}(X_1)$.

Find $\mathrm{P}(X_2 = 0)$ and $\mathrm{P}(X_2 = 1)$ and show that $\mathrm{P}(X_2 = 2) = \frac{1}{6}$.

Hence calculate $\mathrm{E}(X_2)$.

(b) For $n \geqslant 1$, show that

$$\mathrm{P}(X_n = 0) = \sum_{s=0}^{n-1}\frac{\mathrm{P}(X_{n-1} = s)}{s+2}$$

and find a similar expression for $\mathrm{P}(X_n = r)$, for $r = 1, 2, \ldots, n$.

(c) Hence show that $\mathrm{E}(X_n) = \frac{1}{2}\left(1 + \mathrm{E}(X_{n-1})\right)$.

Find an expression for $\mathrm{E}(X_n)$ in terms of n, for $n = 1, 2, \ldots$.

Examiner's report

Most candidates answered all parts of this question well, with many candidates earning full or close to full marks.

In part **(i)**, a small number of candidates erroneously believed that

$$\sum_{r=1}^{m+1}\left(\mathrm{f}(r)\sum_{s=r-1}^{m}\mathrm{g}(s)\right)=\left(\sum_{r=1}^{m+1}\mathrm{f}(r)\right)\times\left(\sum_{s=r-1}^{m}\mathrm{g}(s)\right)$$

and likewise for the second sum. Such attempts earned no credit.

In part **(ii)(a)**, a significant number of candidates did not understand the concept of X_{n+1} being uniformly distributed on $0, 1, \ldots, X_{n+1}$, usually leading to the incorrect values.

In part **(ii)(b)**, a number of candidates gave no justification of the written result, simply writing

$$P(X_n = 0) = \tfrac{1}{2}P(X_{n-1} = 0) + \tfrac{1}{3}P(X_{n-1} = 1) + \cdots + \tfrac{1}{n+1}P(X_{n-1} = n-1) = \sum_{s=0}^{n-1} \frac{P(X_{n-1} = s)}{s = 2}$$

such attempts earned no credit.

In part **(ii)(c)**, most candidates solved this part either by inductively proving that $E(X_n) = 1-2^{-n}$ or by noting that $E(X_n) - 1 = \frac{1}{2}\left[E(X_{n-1}) - 1\right]$ and applying recursion. A smaller number of candidates applied recursion directly to the formula $E(X_n) = \frac{1}{2}\left[E(X_{n-1}) + 1\right]$ leading to a correct solution via geometric series.

Solution

(i) Expanding the first sum gives:

$$\begin{aligned}
&\sum_{r=1}^{m+1}\left(f(r)\sum_{s=r-1}^{m} g(s)\right)\\
={}& f(1)\big[g(0) + g(1) + \cdots + g(m)\big]\\
&\quad + f(2)\big[g(1) + g(2) + \cdots + g(m)\big]\\
&\qquad + f(3)\big[g(2) + g(3) + \cdots + g(m)\big]\\
&\qquad\quad + \cdots\\
&\qquad\qquad + f(m)\big[g(m-1) + g(m)\big]\\
&\qquad\qquad\quad + f(m+1)g(m)\\
={}& g(0)f(1)\\
&\quad + g(1)\big[f(1) + f(2)\big]\\
&\qquad + g(2)\big[f(1) + f(2) + f(3)\big]\\
&\qquad\quad + \cdots\\
&\qquad\qquad + g(m)\big[f(1) + f(2) + \cdots + f(m+1)\big]\\
={}& \sum_{s=0}^{m}\left(g(s)\sum_{r=1}^{s+1} f(r)\right)
\end{aligned}$$

(ii) **(a)** We know that $X_0 = 0$. This means that X_1 takes the values 0 and 1 with probability $\frac{1}{2}$ in each case. This means that $\mathrm{E}(X_1) = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}$. But since this part was "Write down" you could have said $\mathrm{E}(X_1) = \frac{1}{2}$.

If $X_1 = 0$, then X_2 can take values 0 or 1, both with probability $\frac{1}{2}$. If $X_1 = 1$ then X_2 can take values 0, 1 or 2 each with probability $\frac{1}{3}$.

For X_2 we have:

$$\begin{aligned}
\mathrm{P}(X_2 = 0) &= \mathrm{P}(X_1 = 0 \cap X_2 = 0) + \mathrm{P}(X_1 = 1 \cap X_2 = 0) \\
&= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} \\
&= \frac{5}{12} \\
\mathrm{P}(X_2 = 1) &= \mathrm{P}(X_1 = 0 \cap X_2 = 1) + \mathrm{P}(X_1 = 1 \cap X_2 = 1) \\
&= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} \\
&= \frac{5}{12} \\
\mathrm{P}(X_2 = 2) &= \mathrm{P}(X_1 = 1 \cap X_2 = 2) \\
&= \frac{1}{2} \times \frac{1}{3} \\
&= \frac{1}{6}
\end{aligned}$$

Note that the sum of these three probabilities is 1, which is to be expected! This is a useful check just to make sure that you have not made a mistake at this stage.

Hence we have:

$$\mathrm{E}(X_2) = 1 \times \frac{5}{12} + 2 \times \frac{1}{6} = \frac{3}{4}$$

(b) We have:

$$\begin{aligned}
&\mathrm{P}(X_n = 0) \\
=&\mathrm{P}(X_{n-1} = 0 \cap X_n = 0) + \mathrm{P}(X_{n-1} = 1 \cap X_n = 0) + \cdots + \mathrm{P}(X_{n-1} = n-1 \cap X_n = 0) \\
=&\frac{1}{2}\mathrm{P}(X_{n-1} = 0) + \frac{1}{3}\mathrm{P}(X_{n-1} = 1) + \frac{1}{4}\mathrm{P}(X_{n-1} = 2) + \cdots + \frac{1}{n+1}\mathrm{P}(X_{n-1} = n-1) \\
=&\sum_{s=0}^{n-1} \frac{\mathrm{P}(X_{n-1} = s)}{s+2}
\end{aligned}$$

For $\mathrm{P}(X_n = r)$ first note that this is only possible if we have $x_{n-1} \geqslant r - 1$. In a similar way to above we have:

$$\mathrm{P}(X_n = r) = \sum_{s=r-1}^{n-1} \frac{\mathrm{P}(X_{n-1} = s)}{s+2}$$

Note that the above expression does not work when $r = 0$, as this would mean the sum started from $s = -1$.

(c) When finding the expectation we multiply $\mathrm{P}(X_n = 0)$ by 0, so we can ignore this term to get:

$$\begin{aligned}
\mathrm{E}(X_n) &= \sum_{r=1}^{n} r \left(\sum_{s=r-1}^{n-1} \frac{\mathrm{P}(X_{n-1} = s)}{s+2} \right) \\
&= \sum_{s=0}^{n-1} \frac{\mathrm{P}(X_{n-1} = s)}{s+2} \left(\sum_{r=1}^{s+1} r \right) \quad \text{using } \textbf{(i)} \text{ with } m = n-1 \\
&= \sum_{s=0}^{n-1} \frac{\mathrm{P}(X_{n-1} = s)}{s+2} \left(\frac{1}{2}(s+1)(s+2) \right) \quad \text{using } \sum_{k=1}^{n} k = \tfrac{1}{2}n(n+1) \\
&= \frac{1}{2} \sum_{s=0}^{n-1} \mathrm{P}(X_{n-1} = s) \times (s+1) \\
&= \frac{1}{2} \left[\sum_{s=0}^{n-1} \mathrm{P}(X_{n-1} = s) + \sum_{s=0}^{n-1} s\mathrm{P}(X_{n-1} = s) \right] \\
&= \frac{1}{2} \left[1 + \mathrm{E}(X_{n-1}) \right]
\end{aligned}$$

We already have $\mathrm{E}(X_1) = \frac{1}{2}$ and $\mathrm{E}(X_2) = \frac{3}{4}$. Using the recurrence relationship that we have just found we also have:

$$\begin{aligned}
\mathrm{E}(X_3) &= \frac{1}{2}\left[1 + \frac{3}{4}\right] = \frac{7}{8} \\
\mathrm{E}(X_4) &= \frac{1}{2}\left[1 + \frac{7}{8}\right] = \frac{15}{16}
\end{aligned}$$

CONJECTURE: $\mathrm{E}(X_n) = \dfrac{2^n - 1}{2^n}$.

We can see that this is true for $n = 1, 2, 3$ and 4. Assume it is true for $n = k$, so that we have $\mathrm{E}(X_k) = \dfrac{2^k - 1}{2^k}$. Then considering $n = k+1$:

$$\begin{aligned}
\mathrm{E}(X_{k+1}) &= \frac{1}{2}\left[1 + \mathrm{E}(X_k)\right] \\
&= \frac{1}{2}\left[1 + \frac{2^k - 1}{2^k}\right] \\
&= \frac{1}{2} \times \frac{2^k + 2^k - 1}{2^k} \\
&= \frac{2 \times 2^k - 1}{2 \times 2^k} \\
&= \frac{2^{k+1} - 1}{2^{k+1}}
\end{aligned}$$

and so it is true for $n = k+1$. Hence we have $\mathrm{E}(X_n) = \dfrac{2^n - 1}{2^n}$ for $n = 1, 2, \cdots$.