STEP Support Programme

2025 STEP 2 Worked Paper

General comments

These solutions have a lot more words in them than you would expect to see in an exam script and in places I have tried to explain some of my thought processes as I was attempting the questions. What you will not find in these solutions is my crossed out mistakes and wrong turns, but please be assured that they did happen!

You can find the examiners report and mark schemes for this paper from the OCR Website. These are the general comments for the STEP 2 2025 exam from the Examiner's report:

As is commonly the case, the vast majority of candidates focused on the Pure questions in Section A of the paper, with a good number of attempts made on all of those questions. Candidates that attempted the Mechanics questions in Section B generally answered both questions. More candidates attempted Question 11 in Section C than either Mechanics question, but very few attempted Question 12 in that section. There were a large number of good responses seen for all the questions, but a significant number of responses lacked sufficient detail in the presentation, particularly when asked to prove a given result or provide an explanation.

Candidates who did well on this paper generally:

- *gave careful explanations of each step within their solutions*
- *indicated all points of interest on graphs and other diagrams clearly*
- *made clear comments about the approach that needed to be taken, particularly when having to explore a number of cases as part of the solution to a question*
- *used mathematical terminology accurately within their solutions.*

Candidates who did less well on this paper generally:

- *made errors with basic algebraic manipulation, such as incorrect processing of indices*
- *produced sketches of graphs in which significant points were difficult to see clearly because of the chosen scale*
- *skipped important lines within lengthy sections of algebraic reasoning.*

Please send any corrections, comments or suggestions to step@maths.cam.ac.uk.

Question 1

1 The function Min is defined as

$$\text{Min}(a,b) = \begin{cases} a & \text{if } a \leqslant b \\ b & \text{if } a > b\,. \end{cases}$$

(i) Sketch the graph $y = \text{Min}(x^2, 2x)$.

(ii) Solve the equation $2\,\text{Min}(x^2, 2x) = 5x - 3$.

(iii) Solve the equation $\text{Min}(x^2, 2x) + \text{Min}(x^3, 4x) = mx$ in the cases $m = 2$ and $m = 6$.

(iv) Show that $(1, -3)$ is a local maximum point on the curve $y = 2\,\text{Min}(x^2, x^3) - 5x$ and find the other three local maxima and minima on this curve.

Sketch the curve.

Examiner's report

This was a popular question and there were many very good responses seen, with a small number of candidates scoring full marks. Almost all responses included attempts at all parts of the question.

Part **(i)** was generally answered well, although many candidates did not make clear that the gradient of the quadratic section was zero at the origin. Additionally, while most sketches showed two straight line sections for the parts that should show the line $y = 2x$, it was not always clear that these two straight line sections were parts of the same straight line.

Part **(ii)** was well answered, but many candidates omitted the coefficient of 2 when solving the equation and therefore were not able to reach the correct points. Additionally, many solutions did not show sufficient evidence of checking that the solutions fell within the required ranges. There was a small, but significant, number of candidates that struggled to factorise their correct quadratic equation.

Part **(iii)** was well answered, with most candidates able to identify the correct function for each of the ranges and solve the corresponding equations. A common error, however, was to solve the equation $6x = 6x$ either as $x = 0$ or as $x = 0$ or 1, rather than noting that it is valid for any value of x within the relevant range. Some candidates did not combine all of their results from the different ranges correctly but were awarded the marks provided that all of the correct values were seen somewhere within the solution.

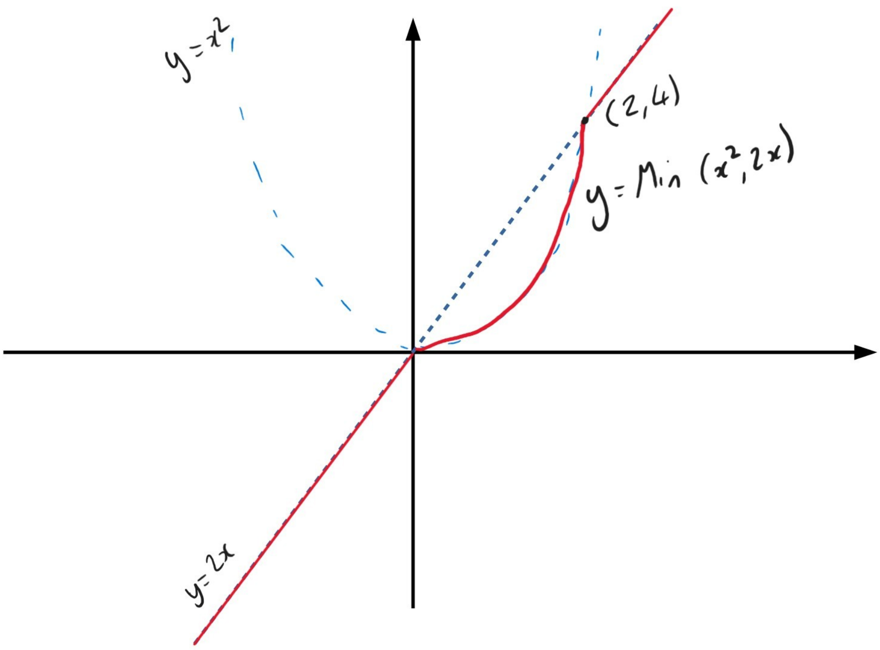
Many candidates struggled with the explanation that $(1, -3)$ is a local maximum of the curve, although there were some very good explanations seen. Candidates were generally good at identifying the other maxima and minima on the curve, although numerical errors, particularly in the simplification of the y coordinates, were common. When sketching the graphs, many candidates were able to draw the quadratic and cubic sections, although there were several examples where the symmetry of the curves was not evident. Many candidates tried to smooth the graph around the point where the two sections join, rather than having a clear change of gradient at that point. There were also several cases where points of significance were not marked on the graph. Almost no candidates attempted to justify the relative positioning of the two minimum points on the graph.

Solution

The function "Min" does exactly what you would expect, it finds the minimum of two values!

(i) Where $x^2 = 2x$ we have $x = 0$ or $x = 2$. Drawing the graphs of $y = x^2$ and $y = 2x$ on the same axes shows that when $x \leqslant 0$ and $x \geqslant 2$ the graph of $y = \text{Min}(x^2, 2x)$ is the same as $y = 2x$ and for $0 \leqslant x \leqslant 2$ the graph of $y = \text{Min}(x^2, 2x)$ is the same as $y = x^2$.

Therefore the graph of $y = \text{Min}(x^2, 2x)$ looks like:



(ii) If $0 \leqslant x \leqslant 2$ we have $\text{Min}(x^2, 2x) = x^2$ and so the equation becomes:

$$\begin{aligned} 2x^2 &= 5x - 3 \\ 2x^2 - 5x + 3 &= 0 \\ (2x-3)(x-1) &= 0 \\ \implies x &= 1, x = \tfrac{3}{2} \end{aligned}$$

Both of the values are in the range $[0, 2]$ so they are both valid solutions.

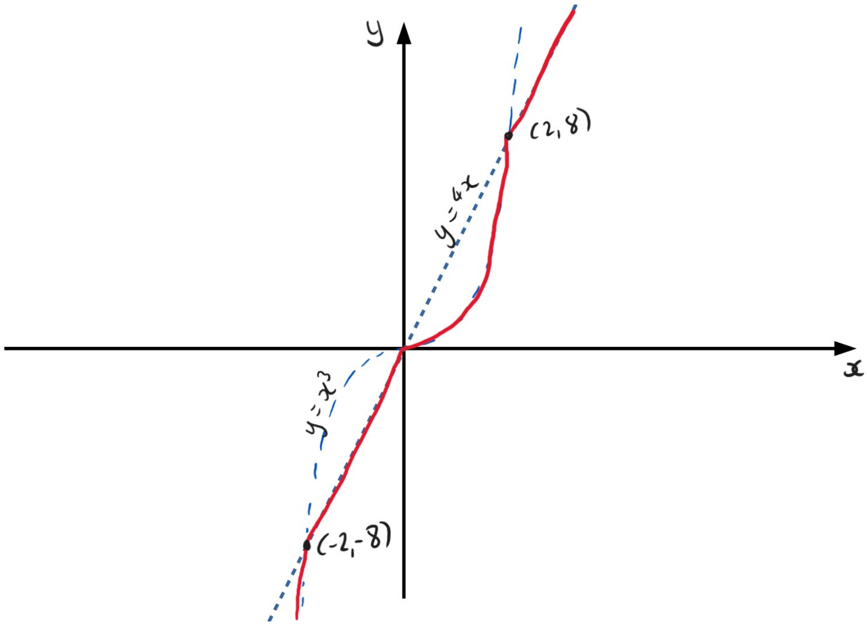
For $x \leqslant 0$ or $x \geqslant 2$ we have:

$$\begin{aligned} 2 \times 2x &= 5x - 3 \\ 4x &= 5x - 3 \\ \implies x &= 3 \end{aligned}$$

and as this is in the range $[2, \infty)$ this is also a solution. Therefore we have three solutions, $x = 1, \frac{3}{2}, 3$.

This Desmos file shows the graphs of $y = 2\text{Min}(x^2, 2x)$ and $y = 5x - 3$.

(iii) The graphs of $y = x^3$ and $y = 4x$ intersect when $x^3 = 4x$, i.e. $x = -2, 0, 2$. Drawing $y = x^3$ and $y = 4x$ on the same axes gives the graph of $y = \text{Min}(x^3, 4x)$ as:



We have:

$$\text{Min}(x^2, 2x) = \begin{cases} x^2 & \text{if } 0 \leqslant x \leqslant 2 \\ 2x & \text{otherwise} \end{cases}$$

and

$$\text{Min}(x^3, 4x) = \begin{cases} x^3 & \text{if } x \leqslant -2 \text{ or } 0 \leqslant x \leqslant 2 \\ 4x & \text{otherwise} \end{cases}$$

Hence we need to consider four different ranges of x. We have:

$$\text{Min}(x^2, 2x) + \text{Min}(x^3, 4x) = \begin{cases} 2x + x^3 & \text{if } x \leqslant -2 \\ 2x + 4x & \text{if } -2 \leqslant x \leqslant 0 \\ x^2 + x^3 & \text{if } 0 \leqslant x \leqslant 2 \\ 2x + 4x & \text{if } x \geqslant 2 \end{cases}$$

Now consider the solution for each value of m.

When $m = 2$ we have:

$$\begin{aligned} x \leqslant -2 &\implies 2x + x^3 = 2x \implies x^3 = 0 && \text{No solution in range} \\ -2 \leqslant x \leqslant 0 &\implies 2x + 4x = 2x \implies x = 0 && \text{Solution } x = 0 \text{ in range} \\ 0 \leqslant x \leqslant 2 &\implies x^2 + x^3 = 2x \implies x^3 + x^2 - 2x = 0 \\ &\implies x(x-1)(x+2) = 0 && \text{Solutions } x = 0, 1 \text{ in range } (x = -2 \text{ is not}) \\ x \geqslant 2 &\implies 2x + 4x = 2x \implies x = 0 && \text{No solution in range} \end{aligned}$$

So when $m = 2$ the solutions are $x = 0$ and $x = 1$.

When $m = 6$ we have:

$$\begin{aligned} x \leqslant -2 &\implies 2x + x^3 = 6x \implies x^3 - 4x = 0 \implies x = 0, x^2 = 4 \\ & \qquad \text{Solution } x = -2 \text{ in range } (x = 0, 2 \text{ are not}) \\ -2 \leqslant x \leqslant 0 &\implies 2x + 4x = 6x \qquad \text{All values in range } -2 \leqslant x \leqslant 0 \text{ are solutions} \\ 0 \leqslant x \leqslant 2 &\implies x^2 + x^3 = 6x \implies x(x-2)(x+3) = 0 \\ & \qquad \text{Solutions } x = 0, 2 \text{ in range } (x = -3 \text{ is not}) \\ x \geqslant 2 &\implies 2x + 4x = 6x \qquad \text{All values in range } x \geqslant 2 \text{ are solutions} \end{aligned}$$

So when $m = 6$ the solution set is $-2 \leqslant x \leqslant 0$ and $x \geqslant 2$.

This Desmos file shows the graph of $y = \text{Min}(x^2, 2x) + \text{Min}(x^3, 4x)$ and $y = mx$, with a slider so that you can change the value of m. Systematic working and clear layout are very important when answering this question!

(iv) Considering $\text{Min}(x^2, x^3)$ we have:

$$\text{Min}(x^2, x^3) = \begin{cases} x^3 & \text{for } x < 1 \\ x^2 & \text{for } x \geqslant 1 \end{cases}$$

We have:

$$\begin{aligned} y &= 2x^3 - 5x && \text{for } x < 1 \\ y &= 2x^2 - 5x && \text{for } x \geqslant 1 \end{aligned}$$

The gradient of the graph is given by:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 6x^2 - 5 && \text{for } x < 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= 4x - 5 && \text{for } x \geqslant 1 \end{aligned}$$

As $x \to 1$ from below we have $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 5 > 0$. As $x \to 1$ from above we have $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x - 5 < 0$. Therefore as x increases across $x = 1$ the gradient changes from being positive to being negative, which means that the point $(1, -3)$ is a local maximum point. Since the gradient does not tend to 0 as $x \to 1$ this maximum point is **not** a stationary point, but is a "sharp" point.

For $x < 1$, solving $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ gives $6x^2 - 5 = 0 \implies x = \pm\sqrt{\frac{5}{6}}$. Note that both of these satisfy $x < 1$. The y coordinates of these points are given by:

$$\begin{aligned} 2\left(\pm\sqrt{\frac{5}{6}}\right)^3 \mp 5\sqrt{\frac{5}{6}} &= \pm\sqrt{\frac{5}{6}} \times \frac{10}{6} \mp 5\sqrt{\frac{5}{6}} \\ &= \sqrt{\frac{5}{6}}\left[\pm\frac{10}{6} \mp 5\right] \\ &= \mp\sqrt{\frac{5}{6}} \times \frac{20}{6} \end{aligned}$$

So the coordinates of these stationary points are $\left(-\sqrt{\frac{5}{6}}, \frac{20}{6}\sqrt{\frac{5}{6}}\right)$ and $\left(\sqrt{\frac{5}{6}}, -\frac{20}{6}\sqrt{\frac{5}{6}}\right)$.

I tried to save time and typing by working out both y coordinates simultaneously, but I don't think this makes the working easy to follow, and it's perhaps safer to consider each one separately.

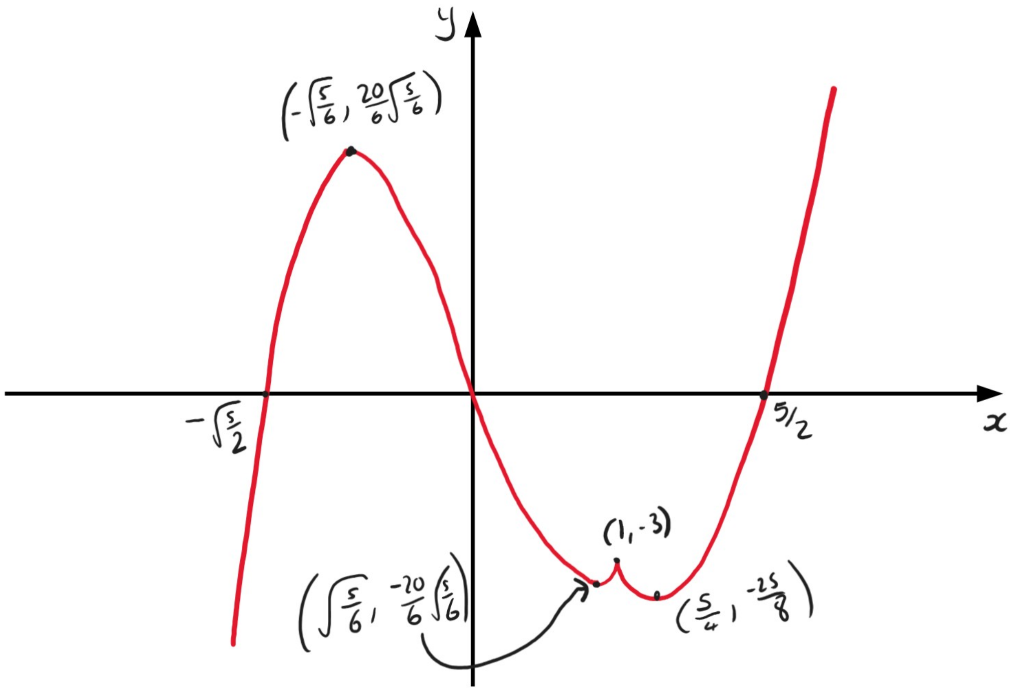
For $x \geqslant 1$ solving $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ gives $4x - 5 = 0 \implies x = \frac{5}{4}$ (which is in range). The y coordinate is:

$$\begin{aligned} 2\left(\frac{5}{4}\right)^2 - 5 \times \frac{5}{4} &= \frac{25}{8} - \frac{25}{4} \\ &= -\frac{25}{8} \end{aligned}$$

and so the other stationary point is at $\left(\frac{5}{4}, -\frac{25}{8}\right)$.

When $x = 0$ we have $y = 0$, and when $y = 0$ for $x < 1$ we have $2x^3 - 5x = 0 \implies x = -\sqrt{\frac{5}{2}}$. For $x > 1$ we have $2x^2 - 5x = 0 \implies x = \frac{5}{2}$.

Putting this all together the graph looks something like:



In this sketch the gradient of the curve close to $(1, -3)$ on my graph is a little too close to infinity for my liking, but the important point is that the graph here should not have zero gradient. Really the gradient should be equal to 1 on the left hand side of $(1, -3)$ and it should be equal to -1 on the right hand side.

For a slightly more accurate graph see Desmos.

Question 2

2 **(i)** **(a)** Show that if the complex number z satisfies the equation

$$z^2 + |z + b| = a,$$

where a and b are real numbers, then z must be either purely real or purely imaginary.

(b) Show that the equation

$$z^2 + \left|z + \tfrac{5}{2}\right| = \tfrac{7}{2}$$

has no purely imaginary roots.

(c) Show that the equation

$$z^2 + \left|z + \tfrac{7}{2}\right| = \tfrac{5}{2}$$

has no purely real roots.

(d) Show that, when $\frac{1}{2} < b < \frac{3}{4}$, the equation

$$z^2 + |z + b| = \tfrac{1}{2}$$

will have at least one purely imaginary root and at least one purely real root.

(ii) Solve the equation

$$z^3 + |z + 2|^2 = 4.$$

Examiner's report

This was a popular question and there was a wide variety in the quality of responses seen, with a small proportion of candidates producing perfect, or close to perfect, solutions.

Part **(i)(a)** was generally completed well, with most candidates explaining the reasoning carefully in their responses. Part **(i)(b)** was also completed well by many candidates, with numerical errors being the main area where marks were lost. In part **(i)(c)**, however, a large number of candidates did not realise there were two cases to be considered and instead only analysed one case. In part **(i)(d)** there was a lot of variation in the quality of responses seen. Those who had successfully completed part **(i)(b)** were often able to demonstrate that there was a purely imaginary root. Those who had successfully completed part **(i)(c)** were often able to make good progress in showing that there must be a purely real root, although many stopped at the point of showing that the discriminant was positive and did not show that at least one of the roots of the quadratic equation lay within the appropriate range. As with part **(i)(c)**, there were a large number of candidates who did not recognise that there were two cases to be explored in this part of the question.

Part **(ii)** was completed well by many candidates, including those who had lost marks in previous parts of the question. Several candidates lost marks through numerical errors or by not explaining clearly enough that all of the solutions had been found in each case.

Solution

(i) **(a)** Let $z = x + \mathrm{i}y$, where x and y are real. We then have:

$$\begin{aligned} z^2 + |z + b| &= a \\ x^2 - y^2 + 2\mathrm{i}xy + |x + \mathrm{i}y + b| &= a \end{aligned}$$

This is a good point to pause before starting to embark on some more complicated algebraic manipulation!

We know that $|x + \mathrm{i}y|$ is real, and so the only imaginary term in the above expression is $-2\mathrm{i}xy$. Therefore we must have $xy = 0$, and so either $x = 0$ or $y = 0$ and z is either purely real or purely imaginary.

(b) If we take $z = \mathrm{i}y$ where y is real then the equation becomes:

$$\begin{aligned} z^2 + \left|z + \tfrac{5}{2}\right| &= \tfrac{7}{2} \\ -y^2 + \left|\mathrm{i}y + \tfrac{5}{2}\right| &= \tfrac{7}{2} \\ \left|\mathrm{i}y + \tfrac{5}{2}\right| &= \tfrac{7}{2} + y^2 \\ y^2 + \tfrac{25}{4} &= y^4 + 7y^2 + \tfrac{49}{4} \\ y^4 + 6y^2 + 6 &= 0 \end{aligned}$$

Solving for y^2 gives:

$$\begin{aligned} y^2 &= \frac{-6 \pm \sqrt{36 - 24}}{2} \\ y^2 &= -3 \pm \sqrt{3} \end{aligned}$$

Both of these values of y^2 are negative, hence there are no purely imaginary roots.

(c) Let $z = x$ where x is real, then we have:

$$\begin{aligned} z^2 + \left|z + \tfrac{7}{2}\right| &= \tfrac{5}{2} \\ x^2 + \left|x + \tfrac{7}{2}\right| &= \tfrac{5}{2} \end{aligned}$$

If $x < -\frac{7}{2}$ then this equation becomes:

$$\begin{aligned} x^2 - \left(x + \tfrac{7}{2}\right) &= \tfrac{5}{2} \\ x^2 - x - 6 &= 0 \\ (x - 3)(x + 2) &= 0 \end{aligned}$$

Which gives solutions $x = 3$ and $x = -2$, but neither of these satisfy $x < -\frac{7}{2}$, so are not valid solutions.
You could also substitute then back into the original equation and show that they do not work, for example if $x = 3$ then the equation becomes:

$$3^2 + \left|3 + \tfrac{7}{2}\right| = \tfrac{5}{2}$$

which is clearly false!

If $x \geqslant -\frac{7}{2}$ then this equation becomes:

$$\begin{aligned} x^2 + \left(x + \tfrac{7}{2}\right) &= \tfrac{5}{2} \\ x^2 + x + 1 &= 0 \end{aligned}$$

Which has discriminant -3 so there are no real solutions for x.
Therefore there are no purely real solutions to the equation.

(d) Let $z = x$ where x is real. The equation becomes:

$$x^2 + |x + b| = \tfrac{1}{2}$$

If $x < -b$ this becomes:

$$\begin{aligned} x^2 - x - b - \tfrac{1}{2} &= 0 \\ x &= \frac{1 \pm \sqrt{1 + 4b + 2}}{2} \end{aligned}$$

Since $\frac{1}{2} < b < \frac{3}{4}$ this will have real solutions, but it's hard to see if these are valid or not (i.e. if they satisfy $x < -b$).

If $x \geqslant -b$ this becomes:

$$\begin{aligned} x^2 + x + b - \tfrac{1}{2} &= 0 \\ x &= \frac{-1 \pm \sqrt{1 - 4b + 2}}{2} \\ x &= \frac{-1 \pm \sqrt{3 - 4b}}{2} \end{aligned}$$

Since $b < \frac{3}{4}$ this will have real solutions. If we look at the positive root then we have:

$$\frac{-1 + \sqrt{3 - 4b}}{2} > -\frac{1}{2}$$

And so if $x \geqslant -b$ then there is a solution such that $x > -\frac{1}{2}$. Since $\frac{1}{2} < b$ we have $-b < -\frac{1}{2}$, and so this **is** a valid solution, and so there exists at least one purely real root.

Let $z = \mathrm{i}y$ where y is real. The equation becomes.

$$\begin{aligned}-y^2 + |\mathrm{i}y + b| &= \tfrac{1}{2}\\ |\mathrm{i}y + b| &= y^2 + \tfrac{1}{2}\\ y^2 + b^2 &= y^4 + y^2 + \tfrac{1}{4}\\ y^4 &= b^2 - \tfrac{1}{4}\end{aligned}$$

Since $b > \frac{1}{2}$ we have $y^4 > 0$ and so there is a real solution for y.

Therefore there is at least one purely imaginary root and at least one purely real root.

(ii) Let $z = x + \mathrm{i}y$. The equation becomes:

$$\begin{aligned}(x + \mathrm{i}y)^3 + |x + \mathrm{i}y + 2|^2 &= 4\\ x^3 + 3\mathrm{i}x^2y - 3xy^2 - \mathrm{i}y^3 + (x+2)^2 + y^2 &= 4\end{aligned}$$

Equating imaginary terms gives:

$$\begin{aligned}3x^2y - y^3 &= 0\\ y(3x^2 - y^2) &= 0\end{aligned}$$

So either $y = 0$ or $3x^2 = y^2$.

If $y = 0$ then we have:

$$\begin{aligned}x^3 + (x+2)^2 &= 4\\ x^3 + x^2 + 4x + 4 &= 4\\ x(x^2 + x + 4) &= 0\end{aligned}$$

The discriminant of $x^2 + x + 4$ is -15, and so the only real solution is $x = 0$.

If $y^2 = 3x^2$ we have:

$$\begin{aligned}x^3 - 3xy^2 + (x+2)^2 + y^2 &= 4\\ x^3 - 3x \times 3x^2 + x^2 + 4x + 4 + 3x^2 &= 4\\ -8x^3 + 4x^2 + 4x &= 0\\ 4x(2x^2 - x - 1) &= 0\\ 4x(2x+1)(x-1) &= 0\\ \implies\ x &= 0, -\tfrac{1}{2}, 1\end{aligned}$$

So the solutions are $z = 0$, $z = 1 \pm \sqrt{3}\mathrm{i}$ or $z = \dfrac{-1 \pm \sqrt{3}\mathrm{i}}{2}$.

Question 3

3 **(i)** Sketch a graph of $y = \dfrac{\ln x}{x}$ for $x > 0$.

(ii) Use your graph to show the following.

(a) $3^{\pi} > \pi^3$

(b) $\left(\frac{9}{4}\right)^{\sqrt{5}} > \sqrt{5}^{\,\frac{9}{4}}$

(iii) Given that $1 < x < 2$, decide, with justification, which is the larger of x^{x+2} or $(x+2)^x$.

(iv) Show that the inequalities $9^{\sqrt{2}} > \sqrt{2}^9$ and $3^{2\sqrt{2}} > \left(2\sqrt{2}\right)^3$ are equivalent. Given that $\mathrm{e}^2 < 8$, decide, with justification, which is the larger of $9^{\sqrt{2}}$ and $\sqrt{2}^9$.

(v) Decide, with justification, which is the larger of $8^{\sqrt[3]{3}}$ and $\sqrt[3]{3}^{\,8}$.

Examiner's report

This was a popular question with many candidates able to make good progress through most parts of the question.

Part **(i)** was answered well by the vast majority of candidates, with the maximum point clearly labelled in most cases. In a small number of cases the asymptotes were not sufficiently clear, although in a small number of cases the behaviour near $x = 0$ was not correct.

Part **(ii)(a)** was generally completed well, with most candidates recognising the relationship between the graph and the required results. In some cases, solutions were presented showing that the given result implied the correct ordering of the numbers, but did not present the logic correctly to show that the ordering of the numbers implies the given result. In part **(ii)(b)** a significant number of candidates did not justify the order of the three numbers within their solution.

Many candidates were able to produce good solutions to part **(iii)**. Some chose to consider a translation of the curve from part **(i)** and looked for the point of intersection between $y = \dfrac{\ln x}{x}$ and $y = \dfrac{\ln(x+2)}{x+2}$ as the method to justify the inequality.

Part **(iv)** was well answered by most candidates who attempted it. Most were able to show the equivalence of the two inequalities, but some only showed the logic in one direction. A large proportion of candidates were then able to see how to apply the equivalence between the two inequalities to determine which of the given values was the larger.

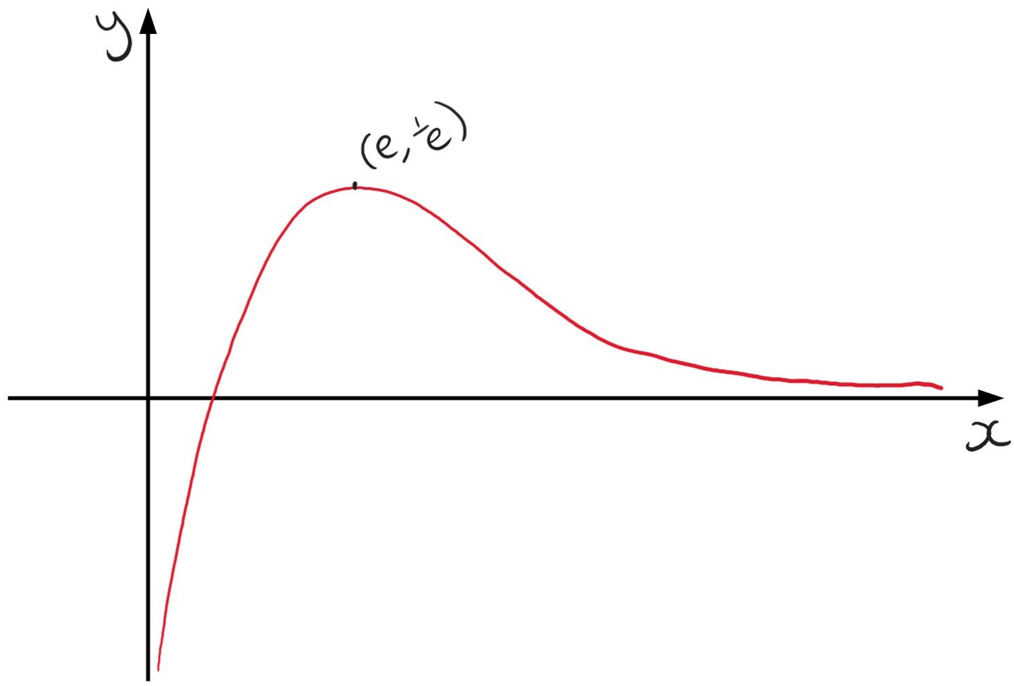
Part **(v)** was found to be very challenging, with some candidates making no written attempt. Those candidates that made progress deduced that there was a need to find an equivalent inequality to allow a similar process to part **(iv)** to be carried out. Several candidates made mistakes with their manipulation of indices within this part of the question. A good proportion of those who identified the equivalent inequality were then able to recognise that an approach similar to part **(iii)** was required to reach the final answer.

Solution

Note that we know that $\mathrm{e} \approx 2.7$.

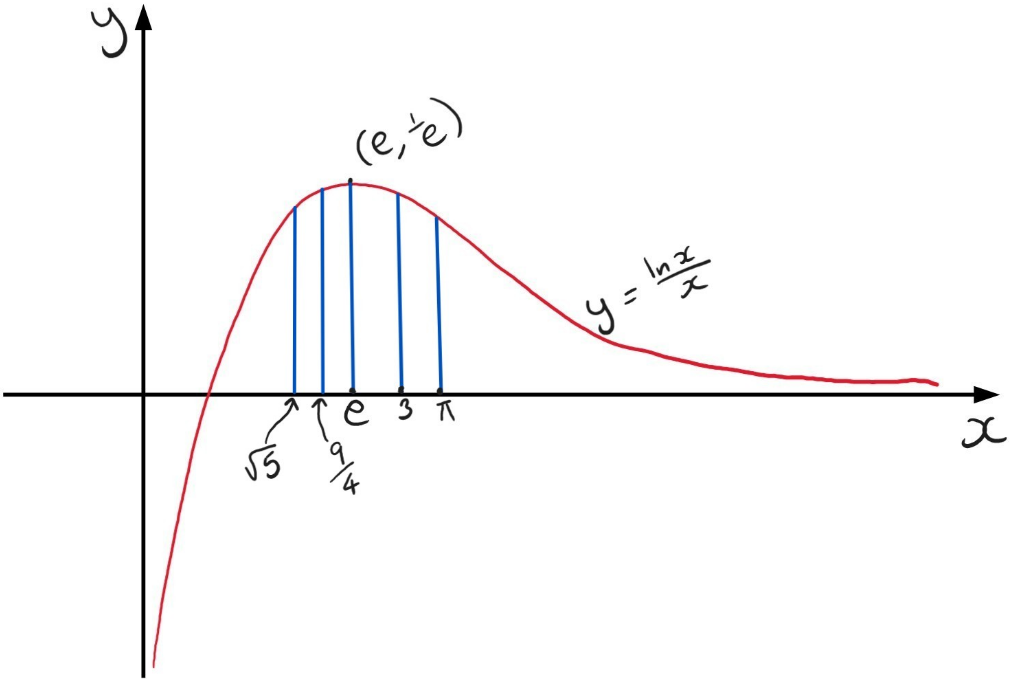
(i) Differentiating gives $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x \times \frac{1}{x} - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$. Therefore there is a stationary point when $x = \mathrm{e}$ and $y = \frac{1}{\mathrm{e}}$.

As $x \to 0_+$, $y \to -\infty$ and as $x \to \infty$, $y \to 0_+$. The graph is only defined for $x > 0$.



(ii) We know that $\mathrm{e} < 3 < \pi$, and that for $x > \mathrm{e}$ the graph is decreasing.

We also know that $\left(\frac{9}{4}\right)^2 = \frac{81}{16} > 5$, so $\sqrt{5} < \frac{9}{4} < \mathrm{e}$, and that for $x < \mathrm{e}$ the graph is increasing.



(a) We have:

$$\frac{\ln 3}{3} > \frac{\ln \pi}{\pi}$$
$$\pi \ln 3 > 3 \ln \pi$$
$$\ln\left(3^{\pi}\right) > \ln\left(\pi^{3}\right)$$

Since $y = \ln x$ is an increasing function we have $3^{\pi} > \pi^{3}$.

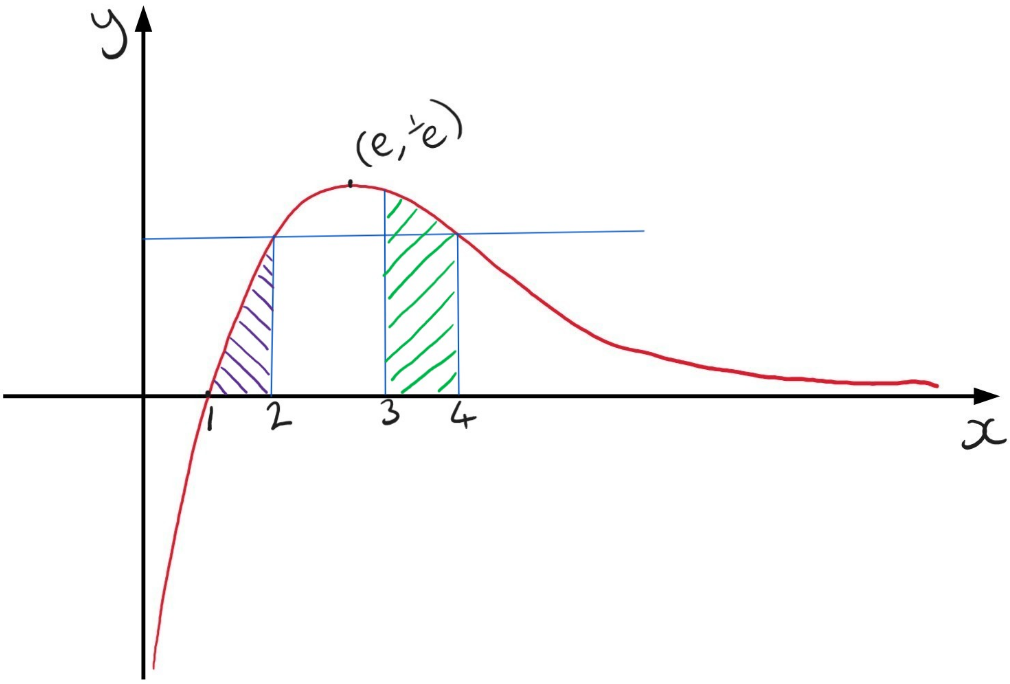
(b) Noting that $\frac{9}{4} = 2.25 < \mathrm{e}$ and $\left(\frac{9}{4}\right)^2 = \frac{81}{16} > 5$, we have $\sqrt{5} < \frac{9}{4} < \mathrm{e}$. Using the graph in part **(i)** gives:

$$\frac{\ln(\sqrt{5})}{\sqrt{5}} < \frac{\ln\left(\frac{9}{4}\right)}{\frac{9}{4}}$$
$$\frac{9}{4}\ln(\sqrt{5}) < \sqrt{5}\ln\left(\frac{9}{4}\right)$$
$$\ln\left(\sqrt{5}^{\frac{9}{4}}\right) < \ln\left[\left(\frac{9}{4}\right)^{\sqrt{5}}\right]$$

Therefore we have $\left(\frac{9}{4}\right)^{\sqrt{5}} > \sqrt{5}^{\frac{9}{4}}$ as required.

(iii) If $1 < x < 2$, then $3 < x + 2 < 4$. We also have $\dfrac{\ln 4}{4} = \dfrac{2\ln 2}{4} = \dfrac{\ln 2}{2}$.

From the diagram below, we can see that if $1 < x < 2$ then we must have $\dfrac{\ln(x+2)}{x+2} > \dfrac{\ln x}{x}$.



This means we have:

$$\frac{\ln(x+2)}{x+2} > \frac{\ln x}{x}$$
$$x\ln(x+2) > (x+2)\ln x$$
$$\ln\left[(x+2)^x\right] > \ln\left[x^{(x+2)}\right]$$

Therefore the largest is $(x+2)^x$.

(iv) We have $9^{\sqrt{2}} = \left(3^2\right)^{\sqrt{2}} = 3^{2\sqrt{2}}$, and $\sqrt{2}^9 = \left(\sqrt{2}^3\right)^3 = \left(2\sqrt{2}\right)^3$.

Since $\mathrm{e}^2 < 8 \implies \mathrm{e} < 2\sqrt{2}$, and we also have $\mathrm{e} < 2\sqrt{2} < 3$, so from the fact that we know that $y = \dfrac{\ln x}{x}$ is decreasing for $x > \mathrm{e}$ we have:

$$\frac{\ln(2\sqrt{2})}{2\sqrt{2}} > \frac{\ln 3}{3}$$
$$\ln\left[(2\sqrt{2})^3\right] > \ln\left[3^{2\sqrt{2}}\right]$$
$$\therefore (2\sqrt{2})^3 > 3^{2\sqrt{2}}$$
$$\implies \sqrt{2}^9 > 9^{\sqrt{2}}$$

(v) We have:

$$8^{\sqrt[3]{3}} = \left(2^3\right)^{\sqrt[3]{3}} = 2^{\left(3\sqrt[3]{3}\right)}$$
$$\sqrt[3]{3}^8 = \left[\left(\sqrt[3]{3}\right)^4\right]^2 = \left(3\sqrt[3]{3}\right)^2$$

So instead we can consider which is the largest out of $2^{\left(3\sqrt[3]{3}\right)}$ and $\left(3\sqrt[3]{3}\right)^2$.

We have $2 < \mathrm{e} < 3\sqrt[3]{3}$, i.e. these values lie on opposites sides of the maximum of the graph $y = \dfrac{\ln x}{x}$, so looking at the method for part **(iii)** might be useful. We know from part **(iii)** that $\dfrac{\ln 2}{2} = \dfrac{\ln 4}{4}$, so if we can determine whether $3\sqrt[3]{3}$ is greater than or less than 4 then we will be able to answer the question.

We have:

$$\begin{aligned}\left[3\sqrt[3]{3}\right]^3 &= 27 \times 3 \\ &= 81 > 4^3\end{aligned}$$

Therefore $3\sqrt[3]{3} > 4$, and so, using the diagram in part **(iii)** we have:

$$\begin{aligned}\frac{\ln 2}{2} &> \frac{\ln\left[3\sqrt[3]{3}\right]}{3\sqrt[3]{3}} \\ 3\sqrt[3]{3}\ln 2 &> 2\ln\left[3\sqrt[3]{3}\right] \\ 2^{3\sqrt[3]{3}} &> \left[3\sqrt[3]{3}\right]^2 \\ \implies 8^{\sqrt[3]{3}} &> \left(\sqrt[3]{3}\right)^8\end{aligned}$$

Question 4

4 Let $\lfloor x \rfloor$ denote the largest integer that satisfies $\lfloor x \rfloor \leqslant x$.
For example, if $x = -4.2$, then $\lfloor x \rfloor = -5$.

(i) Show that, if n is an integer, then $\lfloor x + n \rfloor = \lfloor x \rfloor + n$.

(ii) Let n be a positive integer and define function f_n by

$$\mathrm{f}_n(x) = \lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \ldots + \left\lfloor x + \frac{n-1}{n} \right\rfloor - \lfloor nx \rfloor$$

(a) Show that $\mathrm{f}_n\left(x + \dfrac{1}{n}\right) = \mathrm{f}_n(x)$.

(b) Evaluate $\mathrm{f}_n(t)$ for $0 \leqslant t < \dfrac{1}{n}$.

(c) Hence show that $\mathrm{f}_n(x) \equiv 0$.

(iii) **(a)** Show that $\left\lfloor \dfrac{x}{2} \right\rfloor + \left\lfloor \dfrac{x+1}{2} \right\rfloor = \lfloor x \rfloor$.

(b) Hence, or otherwise, simplify

$$\left\lfloor \frac{x+1}{2} \right\rfloor + \left\lfloor \frac{x+2}{2^2} \right\rfloor + \ldots + \left\lfloor \frac{x+2^k}{2^{k+1}} \right\rfloor + \ldots .$$

Examiner's report

This was a popular question, but candidates often struggled to explain their reasoning with sufficient clarity in many parts.

Almost all candidates seemed to understand why the result in part **(i)** is true, but many were not precise enough in their explanation. The most common approach was to split x as the sum of two values, one of which was an integer, but many candidates did not state that the other part of the number was greater than or equal to 0 and strictly less than 1.

Part **(ii)(a)** was answered well in general, with almost all candidates realising that part **(i)** could be applied here. Solutions were often very well presented for this part. Part **(ii)(b)** also had many good responses, but in some cases the justification that one or more of the terms must be equal to 0 was missing. Many candidates realised that they could combine the results from the previous two parts to obtain this result, but many arguments were incomplete. In particular, some only showed that the result applied for $x \geqslant 0$.

Part **(iii)(a)** was answered well, with most candidates choosing to split the argument into two cases. A significant number realised that this is also a special case of the result in part **(ii)(c)** and obtained the result from the fact that $\mathrm{f}_2\left(\frac{x}{2}\right) = 0$. Many candidates realised in part **(iii)(b)** that the result from **(iii)(a)** could be applied so that the sum could be expressed in a form where most terms cancelled. A common mistake was simply to claim that all but the first term in the sum would cancel and ignore the final term of the partial sum. A few candidates successfully managed to find the complete solution by considering the cases for the different signs of x.

Solution

Note that $\lfloor x \rfloor$ is sometimes called the *floor function*.

(i) We can define the *fractional* part of x as $\{x\}$, which satisfies $x = \lfloor x \rfloor + \{x\}$. A couple of examples:

$$\text{if } x = 3.2 \text{ then } \lfloor x \rfloor = 3 \text{ and } \{x\} = 0.2$$
$$\text{if } x = -4.2 \text{ then } \lfloor x \rfloor = -5 \text{ and } \{x\} = 0.8$$

In all cases we must have $0 \leqslant \{x\} < 1$. We also have $\lfloor x \rfloor = x - \{x\}$. If n is an integer then we have $\{x+n\} = \{x\}$ as adding on an integer to a real number does not affect the fractional part.

Using this definition we have:

$$\begin{aligned}\lfloor x+n \rfloor &= (x+n) - \{x+n\} \\ &= x + n - \{x\} \\ &= \lfloor x \rfloor + n\end{aligned}$$

There are other ways of showing this, but this seemed to be the clearest and easiest to follow.

(ii) **(a)** We have:

$$\begin{aligned}\mathrm{f}_n\left(x+\frac{1}{n}\right) &= \left\lfloor x+\frac{1}{n}\right\rfloor + \left\lfloor x+\frac{2}{n}\right\rfloor + \cdots + \left\lfloor x+\frac{n-1}{n}\right\rfloor + \left\lfloor x+\frac{n}{n}\right\rfloor - \left\lfloor n\left(x+\frac{1}{n}\right)\right\rfloor \\ &= \left\lfloor x+\frac{1}{n}\right\rfloor + \left\lfloor x+\frac{2}{n}\right\rfloor + \cdots + \left\lfloor x+\frac{n-1}{n}\right\rfloor + \lfloor x+1 \rfloor - \lfloor nx+1 \rfloor \\ &= \left\lfloor x+\frac{1}{n}\right\rfloor + \left\lfloor x+\frac{2}{n}\right\rfloor + \cdots + \left\lfloor x+\frac{n-1}{n}\right\rfloor + \lfloor x \rfloor + \cancel{1} - \lfloor nx \rfloor - \cancel{1} \\ &= \mathrm{f}_n(x)\end{aligned}$$

as required. Note that from part **(i)** we have $\lfloor x+1 \rfloor = \lfloor x \rfloor + 1$.

(b) We have:

$$f_n(t) = \lfloor t \rfloor + \left\lfloor t + \frac{1}{n} \right\rfloor + \left\lfloor t + \frac{2}{n} \right\rfloor + \cdots + \left\lfloor t + \frac{n-1}{n} \right\rfloor - \lfloor nt \rfloor$$

Since $0 \leqslant t < \dfrac{1}{n}$ we have $0 \leqslant nt < 1$ and $\dfrac{k}{n} \leqslant t + \dfrac{k}{n} < \dfrac{k+1}{n}$. If $0 \leqslant k \leqslant n-1$ then we have $0 \leqslant t + \dfrac{k}{n} < 1$. Therefore all of the inputs into the floor functions in the expression for $f_n(t)$ lie in the range $[0, 1)$[1], and so we have $f_n(t) = 0$.

(c) From part **(a)** we have $f_n(y) = f_n\left(y + \dfrac{1}{n}\right) = f_n\left(y + \dfrac{2}{n}\right) = f_n\left(y + \dfrac{k}{n}\right)$ for any integer $k \geqslant 0$ and conversely we also have $f_n(y) = f_n\left(y - \dfrac{1}{n}\right) = f_n\left(y - \dfrac{2}{n}\right) = f_n\left(y - \dfrac{k}{n}\right)$.

Therefore for all x we can add or subtract multiples of $\dfrac{1}{n}$ to get

$$f_n(x) = f_n(t) \text{ where } 0 \leqslant t < \frac{1}{n}$$

and so for all x we have $f_n(x) = 0$ i.e. we have $f_n(x) \equiv 0$.

(iii) **(a)** Let $\dfrac{x}{2} = t$ where $t = \lfloor t \rfloor + \{t\}$, and by definition we have $\lfloor t \rfloor$ is an integer and $0 \leqslant \{t\} < 1$. This means that we can take $x = 2k + \delta$ where k is an integer and $0 \leqslant \delta < 2$. We need to consider two different cases.

If $0 \leqslant \delta < 1$ then we have:

$$\begin{aligned} \lfloor x \rfloor &= 2k \\ \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+1}{2} \right\rfloor &= \left\lfloor k + \frac{\delta}{2} \right\rfloor + \left\lfloor k + \frac{\delta+1}{2} \right\rfloor \\ &= k + k \quad \text{as } \delta + 1 < 2 \\ &= \lfloor x \rfloor \end{aligned}$$

If $1 \leqslant \delta < 2$ then we have:

$$\begin{aligned} \lfloor x \rfloor &= 2k + 1 \\ \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+1}{2} \right\rfloor &= \left\lfloor k + \frac{\delta}{2} \right\rfloor + \left\lfloor k + \frac{\delta+1}{2} \right\rfloor \\ &= k + (k+1) \quad \text{as } 2 \leqslant \delta + 1 < 3 \\ &= \lfloor x \rfloor \end{aligned}$$

Therefore we have $\left\lfloor \dfrac{x}{2} \right\rfloor + \left\lfloor \dfrac{x+1}{2} \right\rfloor = \lfloor x \rfloor$ for all possible values of x.

[1] Note that $[0, 1)$ means the interval between 0 and 1, including 0 but not including 1.

(b) We have $\dfrac{x+2^k}{2^{k+1}} = \dfrac{\left(\frac{x}{2^k}\right)+1}{2}$ and so:

$$\begin{aligned}\left\lfloor \frac{x+2^k}{2^{k+1}} \right\rfloor &= \left\lfloor \frac{\left(\frac{x}{2^k}\right)+1}{2} \right\rfloor \\ &= \left\lfloor \frac{x}{2^k} \right\rfloor - \left\lfloor \frac{\left(\frac{x}{2^k}\right)}{2} \right\rfloor \quad \text{using part } \textbf{(iii)(a)} \\ &= \left\lfloor \frac{x}{2^k} \right\rfloor - \left\lfloor \frac{x}{2^{k+1}} \right\rfloor\end{aligned}$$

Consider the finite sum $\mathrm{g}_n(x)$ where:

$$\begin{aligned}\mathrm{g}_n(x) &= \left\lfloor \frac{x+1}{2} \right\rfloor + \left\lfloor \frac{x+2}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{x+2^n}{2^{n+1}} \right\rfloor \\ &= \left(\lfloor x \rfloor - \left\lfloor \frac{x}{2} \right\rfloor\right) + \left(\left\lfloor \frac{x}{2} \right\rfloor - \left\lfloor \frac{x}{2^2} \right\rfloor\right) + \left(\left\lfloor \frac{x}{2^2} \right\rfloor - \left\lfloor \frac{x}{2^3} \right\rfloor\right) + \cdots + \left(\left\lfloor \frac{x}{2^n} \right\rfloor - \left\lfloor \frac{x}{2^{n+1}} \right\rfloor\right) \\ &= \lfloor x \rfloor - \left\lfloor \frac{x}{2^{n+1}} \right\rfloor\end{aligned}$$

If $x \geqslant 0$ then for large enough n we have $0 \leqslant x < 2^n$ and so $\left\lfloor \dfrac{x}{2^{n+1}} \right\rfloor = 0$.

If $x < 0$ then for large enough n we have $-2^n < x \leqslant 0$ and so $\left\lfloor \dfrac{x}{2^{n+1}} \right\rfloor = -1$.

Therefore the sum is equal to:

$$\begin{cases} \lfloor x \rfloor & \text{for } x \geqslant 0 \\ \lfloor x \rfloor + 1 & \text{for } x < 0 \end{cases}$$

Question 5

5 *You need not consider the convergence of the improper integrals in this question.*

(i) Use the substitution $x = u^{-1}$ to show that

$$\int_0^{\infty} \frac{\sqrt{x}-1}{\sqrt{x(x^3+1)}}\,\mathrm{d}x = 0\,.$$

(ii) Use the substitution $x = u^{-2}$ to show that

$$\int_0^{\infty} \frac{1}{\sqrt{x^3+1}}\,\mathrm{d}x = 2\int_0^{\infty} \frac{1}{\sqrt{x^6+1}}\,\mathrm{d}x\,.$$

(iii) Find, in terms of p and s, a value of r for which

$$\int_0^{\infty} \frac{x^r-1}{\sqrt{x^s(x^p+1)}}\,\mathrm{d}x = 0,$$

given that p and s are fixed values for which the required integrals converge.

(iv) Show that, for any positive value of k, it is possible to find values of p and q for which

$$\int_0^{\infty} \frac{1}{\sqrt{x^p+1}}\,\mathrm{d}x = k\int_0^{\infty} \frac{1}{\sqrt{x^q+1}}\,\mathrm{d}x\,.$$

Examiner's report

This was the second most popular question after Question 1 and was attempted by the vast majority of candidates. In general, solutions were very good, particularly for the first two parts.

Parts **(i)** and **(ii)** were answered well, with candidates generally showing a good level of proficiency with completing the given substitutions. In part **(i)** a small number of candidates did not give enough detail in their method when dealing with the limits of the integral following the substitution. Most recognised that the substitution could be used to show that $I = -I$ and produced clear explanations of this. Solutions to part **(ii)** were often fully correct.

Those candidates who were able to identify the correct substitution were often successful in solving part **(iii)**, although in some cases errors were made with the indices when simplifying the expression. Some candidates attempted substitutions which did not allow them to make any significant progress on solving this part of the question.

Part **(iv)** was generally answered more successfully than part **(iii)** with most candidates able to identify the correct substitution to be made. Some candidates started with a more general substitution, from which the form that was needed was deduced. The substitution was again completed successfully by most candidates who reached this part, and the most complete responses noted that the change to the limits would be valid for any of the appropriate values for k. Having completed the substitution, many were able to identify a possible pair of values for p and q. Those who tried to argue that such a pair must exist often did not explain their reasoning clearly enough.

Solution

(i) If $x = u^{-1}$ then we have $\dfrac{\mathrm{d}x}{\mathrm{d}u} = -u^{-2}$. The integral becomes:

$$\begin{aligned}
\int_0^\infty \frac{\sqrt{x}-1}{\sqrt{x(x^3+1)}}\,\mathrm{d}x &= \int_\infty^0 \frac{u^{-\frac{1}{2}}-1}{\sqrt{u^{-1}(u^{-3}+1)}} \times -u^{-2}\,\mathrm{d}u \\
&= \int_0^\infty \frac{1-u^{\frac{1}{2}}}{u^{\frac{1}{2}}u^2\sqrt{u^{-1}(u^{-3}+1)}}\,\mathrm{d}u \\
&= \int_0^\infty \frac{1-u^{\frac{1}{2}}}{u^{\frac{1}{2}}\sqrt{(1+u^3)}}\,\mathrm{d}u \\
&= \int_0^\infty \frac{1-\sqrt{u}}{\sqrt{u(1+u^3)}}\,\mathrm{d}u \\
&= \int_0^\infty \frac{1-\sqrt{x}}{\sqrt{x(1+x^3)}}\,\mathrm{d}x \\
&= -\int_0^\infty \frac{\sqrt{x}-1}{\sqrt{x(1+x^3)}}\,\mathrm{d}x
\end{aligned}$$

Therefore the integral satisfies $I = -I$, and so we have $\displaystyle\int_0^\infty \frac{\sqrt{x}-1}{\sqrt{x(1+x^3)}}\,\mathrm{d}x = 0$ as required.

(ii) If $x = u^{-2}$ then $\dfrac{\mathrm{d}x}{\mathrm{d}u} = -2u^{-3}$. We have:

$$\begin{aligned}
\int_0^\infty \frac{1}{\sqrt{x^3+1}}\mathrm{d}x &= \int_\infty^0 \frac{1}{\sqrt{u^{-6}+1}} \times -2u^{-3}\,\mathrm{d}u \\
&= 2\int_0^\infty \frac{1}{u^3\sqrt{u^{-6}+1}}\,\mathrm{d}u \\
&= 2\int_0^\infty \frac{1}{\sqrt{1+u^6}}\,\mathrm{d}u \\
&= 2\int_0^\infty \frac{1}{\sqrt{x^6+1}}\,\mathrm{d}x
\end{aligned}$$

(iii) This looks it might get a little messy (in fact I went on to do part **(iv)** before coming back to this part!).

Let's try $x = u^{-1}$ and see what happens. We have:

$$\begin{aligned}\int_0^{\infty} \frac{x^r - 1}{\sqrt{x^s(x^p+1)}}\,\mathrm{d}x &= \int_{\infty}^{0} \frac{u^{-r}-1}{\sqrt{u^{-s}(u^{-p}+1)}} \times -u^{-2}\,\mathrm{d}u \\ &= \int_0^{\infty} \frac{u^{-r}-1}{\sqrt{u^{-s}(u^{-p}+1)}} \times u^{-2}\,\mathrm{d}u \\ &= \int_0^{\infty} \frac{1-u^r}{u^r u^2\sqrt{u^{-s}(u^{-p}+1)}}\,\mathrm{d}u \\ &= \int_0^{\infty} \frac{1-u^r}{\sqrt{u^{2r+4-s-p}+u^{2r+4-s}}}\,\mathrm{d}u\end{aligned}$$

We want this to be the negative of the original integral, so we want $2r+4-s = s+p$ and $2r+4-s-p = s$. These are both satisfied by $r = s + \frac{p}{2} - 2$.

Wasn't so bad after all!

(iv) Let $x = u^{-k}$, which gives $\dfrac{\mathrm{d}x}{\mathrm{d}u} = -ku^{-(k+1)}$. Using this substitution gives:

$$\begin{aligned}\int_0^{\infty} \frac{1}{\sqrt{x^p+1}}\,\mathrm{d}x &= \int_{\infty}^{0} \frac{1}{\sqrt{u^{-kp}+1}} \times -ku^{-(k+1)}\,\mathrm{d}u \\ &= k\int_0^{\infty} \frac{1}{u^{k+1}\sqrt{u^{-kp}+1}}\,\mathrm{d}u \\ &= k\int_0^{\infty} \frac{1}{\sqrt{u^{-kp+2k+2}+u^{2k+2}}}\,\mathrm{d}u\end{aligned}$$

Therefore we need $-kp+2k+2 = 0 \implies p = 2+\frac{2}{k}$ and $2k+2 = q$.

It's useful to check that these answers agree with the values in part **(ii)**, i.e. if $k=2$, $p=3$ and $q=6$ then $2+\frac{2}{k} = 3 = p$ and $2k+2 = 6 = q$. It's always a good idea to check your answers if the opportunity arises.

Question 6

6 **(i)** The circle $x^2 + (y - a)^2 = r^2$ touches the parabola $2ky = x^2$, where $k > 0$, tangentially at two points. Show that $r^2 = k(2a - k)$.

Show further that if $r^2 = k(2a-k)$ and $a > k > 0$, then the circle $x^2+(y-a)^2 = r^2$ touches the parabola $2ky = x^2$ tangentially at two points.

(ii) The lines $y = c \pm x$ are tangents to the circle $x^2 + (y - a)^2 = r^2$. Find r^2, and the coordinates of the points of contact, in terms of a and c.

(iii) C_1 and C_2 are circles with equations $x^2 + (y - a_1)^2 = r_1^2$ and $x^2 + (y - a_2)^2 = r_2^2$ respectively, where $a_1 \neq a_2$ and $r_1 \neq r_2$.

Each circle touches the parabola $2ky = x^2$ tangentially at two points and the lines $y = c \pm x$ are tangents to both circles.

(a) Show that $a_1 + a_2 = 2c + 4k$ and that $a_1^2 + a_2^2 = 2c^2 + 16kc + 12k^2$.

(b) The circle $x^2 + (y - d)^2 = p^2$ passes through the four points of tangency of the lines $y = c \pm x$ to the two circles, C_1 and C_2. Find d and p^2 in terms of k and c.

(c) Show that the circle $x^2 + (y - d)^2 = p^2$ also touches the parabola $2ky = x^2$ tangentially at two points.

Examiner's report

This was one of the less popular questions from the Pure section of the paper, but still received many attempts. This question was found to be challenging and few candidates gained many of the marks.

In part **(i)** almost all solutions attempted to use a calculus or discriminant argument. When arguing based on the discriminant, candidates often did not explain sufficiently clearly how the value of the discriminant related to tangency. Many candidates only solved this part of the question in one direction, not realising that the converse required a separate argument.

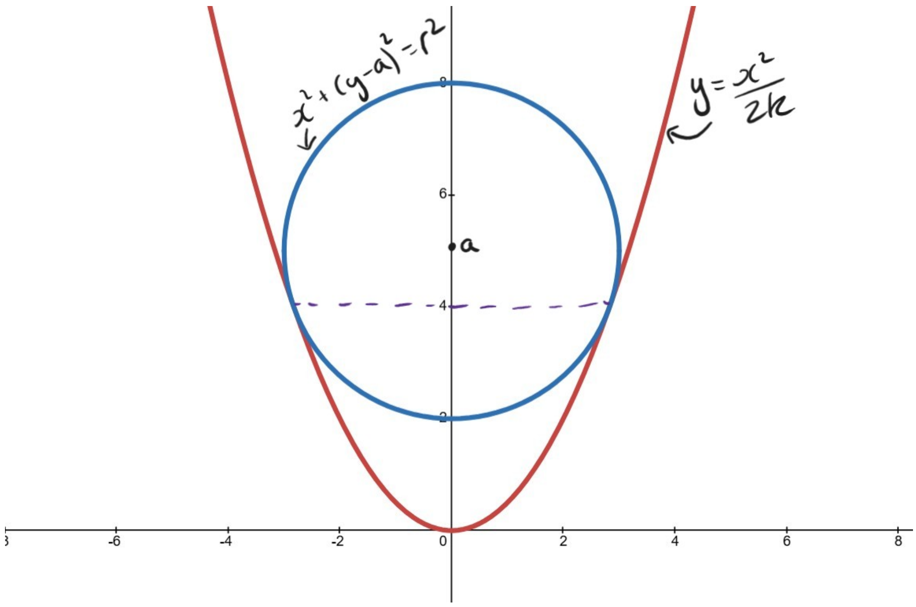
Part **(ii)** was generally completed more successfully than part **(i)**, but responses frequently included algebraic mistakes or did not appreciate the number of solutions that needed to be found.

Very few candidates attempted part **(iii)**. Part **(iii)(a)** was done well by many of those who attempted it. In part **(iii)(b)** there were again a number of algebraic mistakes seen. Most candidates who attempted part **(iii)(c)** related it back to part **(i)**, but most did not make any justification beyond the algebraic manipulation.

Solution

There are a lot of equations in this solution, and I had to be slightly creative with my labelling.

(i) It's a good idea to draw a sketch to show the situation when the circle is tangential to the parabola.



From the sketch we can see that when the curves meet tangentially at two points we only have one value for y for which the curves meet (the graphs are symmetric in x so there will be two x values for this one y value).

Where the curves meet we have:

$$\begin{aligned} x^2 + (y-a)^2 &= r^2 \\ 2ky + (y-a)^2 &= r^2 \qquad (*) \\ y^2 + (2k-2a)y + a^2 - r^2 &= 0 \end{aligned}$$

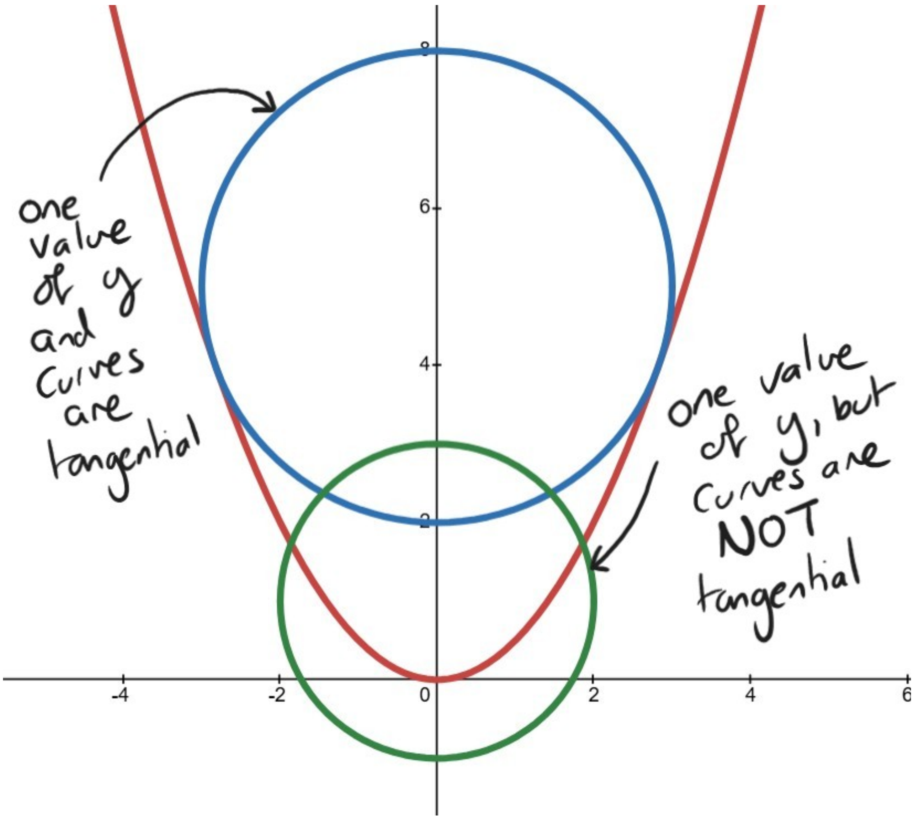
For there to be a repeated root for y we need:

$$\begin{aligned} 4(k-a)^2 - 4(a^2-r^2) &= 0 \\ k^2 - 2ka + \cancel{a^2} - \cancel{a^2} + r^2 &= 0 \\ r^2 &= 2ka - k^2 \\ \implies r^2 &= k(2a-k) \end{aligned}$$

We cannot simply reverse all the logic, as whilst we have:

$$\text{curves are tangential} \implies \text{repeated root for } y \text{ in } (*)$$

it is not necessarily the case that a repeated root for y means that the curves will be tangential, as shown in the sketch below.



So for the second direction of implication we need to do a little more work. We know that if $r^2 = k(2a - k)$ there will only be one value of y, so exactly two intersections (unless they meet only at the point $(0, 0)$).

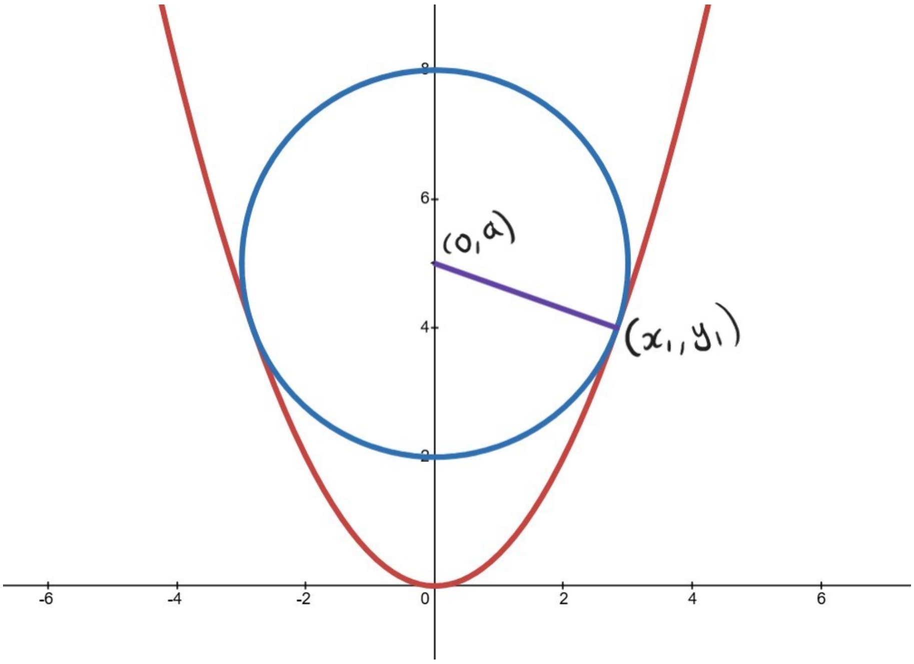
If the curves meet at $(0, 0)$ then we must have $a = r$ (noting that $a > 0$, so we cannot have $a = -r$). If we have $a = r$ then the condition on r^2 gives:

$$\begin{aligned} r^2 &= k(2a - k) \\ a^2 &= k(2a - k) \\ a^2 - 2ak + k^2 &= 0 \\ (a - k)^2 &= 0 \end{aligned}$$

which gives $a = k$, but we are told that $a > k$. Therefore the curves do not intersect at $(0, 0)$, and so there are two intersections (each with the same value of y). It remains to show that the tangents to the curves are the same at these points.

Let the point of intersection with positive x coordinate be at (x_1, y_1). The gradient of the parabola $y = \dfrac{x^2}{2k}$ at this point is equal to $\dfrac{x_1}{k}$. For this to be defined then we need $k \neq 0$ (which is true as we have $k > 0$).

We could use implicit differentiation to find the gradient of the circle at this point, or we could use some geometry.



The gradient of the radius connecting $(0, a)$ and (x_1, y_1) is $\dfrac{y_1 - a}{x_1}$, so the gradient of the tangent to the circle at this point is $\dfrac{x_1}{a - y_1}$.

Therefore if we can show that $k = a - y_1$ then the gradients of the tangents will be equal.

If we have $r^2 = k(2a - k)$ then $(*)$ becomes:

$$\begin{aligned}
2ky + (y-a)^2 &= r^2 \\
2ky + (y-a)^2 &= k(2a-k) \\
y^2 + 2(k-a)y + a^2 - 2ak + k^2 &= 0 \\
y^2 + 2(k-a)y + (k-a)^2 &= 0 \\
(y + k - a)^2 &= 0
\end{aligned}$$

Therefore we have $y_1 = a - k$, and so $x_1^2 = 2k(a - k)$. For this to have two real solutions we need $a > k$.

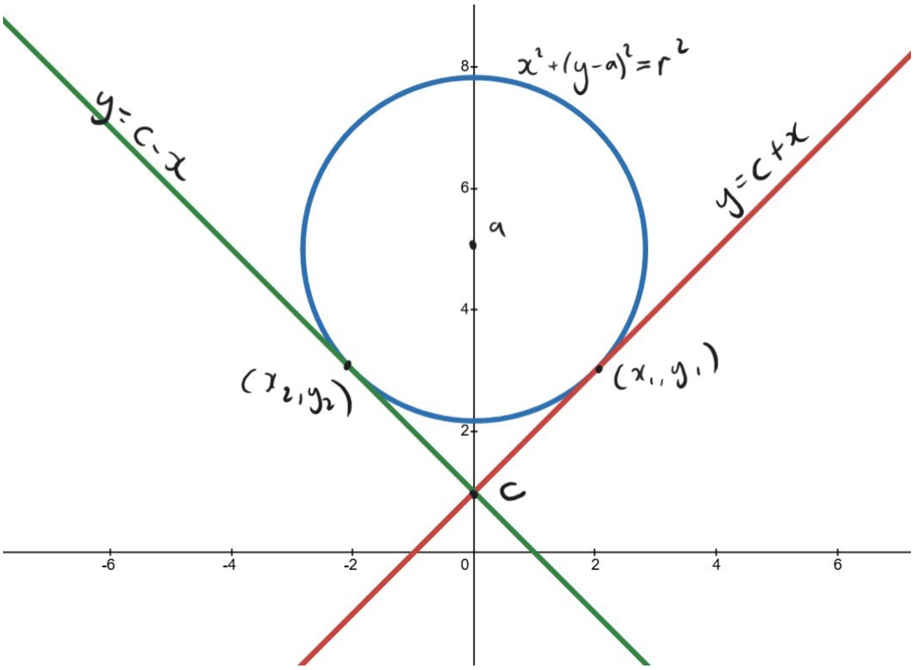
Hence if $r^2 = k(2a - k)$ and $a > k > 0$ then the curves meet at two points satisfying $y_1 = a - k \implies k = a - y_1$. Since the gradients of the tangents are $\dfrac{x_1}{k}$ and $\dfrac{x_1}{a - y_1}$ the gradients of the tangents are equal at this point, so the curves touch tangentially at two points.

There is probably more explanation than needed here, and I could have used the gradients of the tangents for the first direction of implication rather than the discriminant argument which might have been quicker.

Note that for the second direction of implication there are more restrictions given. This implies that you might need to use a different argument for the second implication.

(ii) Using the repeated root argument does not seem to be very friendly in the case.

The situation described looks like this:



We can see from this diagram that in order to have two tangential points of intersection we must have $c < a - r$, or $c > a + r$ in the case that the tangents meet "above" the circle.

From the previous part we know that the gradient of the circle at (x_1, y_1) is given by $\dfrac{x_1}{a - y_1}$. Where this has gradient 1 we have $x_1 = a - y_1$. If this point also lies on the line $y = c + x$ then we have $y_1 = c + x_1$.

Solving these simultaneously we can get one point of intersection at $\left(\dfrac{a-c}{2}, \dfrac{a+c}{2}\right)$. By symmetry the other point of intersection is at $\left(\dfrac{c-a}{2}, \dfrac{a+c}{2}\right)$. Note that these seem reasonable results when compared to the diagram.

Substituting into the equation of the circle we have:

$$
\begin{aligned}
x^2 + (y-a)^2 &= r^2 \\
\left(\frac{a-c}{2}\right)^2 + \left(\frac{a+c}{2} - a\right)^2 &= r^2 \\
\left(\frac{a-c}{2}\right)^2 + \left(\frac{a+c-2a}{2}\right)^2 &= r^2 \\
2\left(\frac{a-c}{2}\right)^2 &= r^2 \\
\implies r^2 &= \tfrac{1}{2}(a-c)^2
\end{aligned}
$$

(iii) **(a)** At this stage trying to sketch the graphs is not so helpful (and is getting more difficult).

Using parts **(i)** and **(ii)** we have:

$$r_1^2 = k(2a_1 - k) \quad (1)$$
$$r_1^2 = \tfrac{1}{2}(a_1 - c)^2 \quad (2)$$
$$r_2^2 = k(2a_2 - k) \quad (3)$$
$$r_2^2 = \tfrac{1}{2}(a_2 - c)^2 \quad (4)$$

Equating (1) and (2) gives:

$$\tfrac{1}{2}(a_1 - c)^2 = k(2a_1 - k)$$
$$\Longrightarrow \; a_1^2 - 2a_1c + c^2 = 4ka_1 - 2k^2 \quad (\dagger)$$

Similarly:

$$a_2^2 - 2a_2c + c^2 = 4ka_2 - 2k^2 \quad (\ddagger)$$

Subtracting (‡) from (†) gives:

$$(a_1^2 - a_2^2) - 2c(a_1 - a_2) = 4k(a_1 - a_2)$$

Since $a_1 \neq a_2$ we can divide throughout by $(a_1 - a_2)$ to get:

$$a_1 + a_2 - 2c = 4k$$

and so $a_1 + a_2 = 2c + 4k$ as required.

Adding (†) and (‡) gives:

$$\begin{aligned}
(a_1^2 - 2a_1c + c^2) + (a_2^2 - 2a_2c + c^2) &= (4ka_1 - 2k^2) + (4ka_2 - 2k^2) \\
a_1^2 + a_2^2 - 2c(a_1 + a_2) + 2c^2 &= 4k(a_1 + a_2) - 4k^2 \\
a_1^2 + a_2^2 + 2c^2 &= (4k + 2c)(a_1 + a_2) - 4k^2 \\
a_1^2 + a_2^2 + 2c^2 &= (4k + 2c)^2 - 4k^2 \text{ using the previous result} \\
a_1^2 + a_2^2 + 2c^2 &= 16k^2 + 16kc + 4c^2 - 4k^2 \\
a_1^2 + a_2^2 &= 12k^2 + 16kc + 2c^2 \text{ as required.}
\end{aligned}$$

(b) We know that the circle $x^2 + (y - d)^2 = p^2$ passes through the two points where C_1 is tangent to $y = c \pm x$ and the two points where C_2 is a tangent to $y = c \pm x$. Therefore we know that the points $\left(\pm\frac{a_1 - c}{2}, \frac{a_1 + c}{2}\right)$ and $\left(\pm\frac{a_2 - c}{2}, \frac{a_2 + c}{2}\right)$ lie on the circle. Substituting these into the equation for the circle we have:

$$\left(\frac{a_1 - c}{2}\right)^2 + \left(\frac{a_1 + c}{2} - d\right)^2 = p^2 \quad (\Delta)$$

$$\text{and} \quad \left(\frac{a_2 - c}{2}\right)^2 + \left(\frac{a_2 + c}{2} - d\right)^2 = p^2 \quad (\Delta\Delta)$$

Subtracting gives:

$$\left(\frac{a_1-c}{2}\right)^2-\left(\frac{a_2-c}{2}\right)^2=\left(\frac{a_2+c}{2}-d\right)^2-\left(\frac{a_1+c}{2}-d\right)^2$$

Using difference of two squares with the left hand side gives:

$$\begin{aligned}&\left[\left(\frac{a_1-c}{2}\right)+\left(\frac{a_2-c}{2}\right)\right]\left[\left(\frac{a_1-c}{2}\right)-\left(\frac{a_2-c}{2}\right)\right]\\=&\left[\frac{a_1+a_2-2c}{2}\right]\left[\frac{a_1-a_2}{2}\right]\end{aligned}$$

and on the right hand side we have:

$$\begin{aligned}&\left[\left(\frac{a_2+c}{2}-d\right)+\left(\frac{a_1+c}{2}-d\right)\right]\left[\left(\frac{a_2+c}{2}-d\right)-\left(\frac{a_1+c}{2}-d\right)\right]\\=&\left[\frac{a_2+a_1+2c-4d}{2}\right]\left[\frac{a_2-a_1}{2}\right]\end{aligned}$$

Cancelling the common factor of $\left[\frac{a_1-a_2}{2}\right]$ (which is fine as we are told that $a_1\neq a-2$) gives:

$$\begin{aligned}\left[\frac{a_1+a_2-2c}{2}\right]&=-\left[\frac{a_2+a_1+2c-4d}{2}\right]\\a_1+a_2-2c&=-a_1-a_2-2c+4d\\\implies d&=\frac{a_1+a_2}{2}\\\implies d&=c+2k \text{ using part } \textbf{(iii)(a)}\end{aligned}$$

Substituting for d into (Δ) gives:

$$\begin{aligned}p^2&=\left(\frac{a_1-c}{2}\right)^2+\left(\frac{a_1+c}{2}-c-2k\right)^2\\&=\left(\frac{a_1-c}{2}\right)^2+\left(\frac{a_1-c}{2}-2k\right)^2\\&=2\left(\frac{a_1-c}{2}\right)^2-2k(a_1-c)+4k^2\end{aligned}$$

Similarly, $(\Delta\Delta)$ gives:

$$p^2=2\left(\frac{a_2-c}{2}\right)^2-2k(a_2-c)+4k^2$$

Adding these two gives:

$$\begin{aligned}
2p^2 &= 2\left(\frac{a_1 - c}{2}\right)^2 + 2\left(\frac{a_2 - c}{2}\right)^2 - 2k(a_1 + a_2 - 2c) + 8k^2 \\
&= \left(\frac{a_1^2 - 2a_1c + c^2}{2}\right) + \left(\frac{a_2^2 - 2a_2c + c^2}{2}\right) - \cancel{2k(4k)} + \cancel{8k^2} \\
&= \frac{a_1^2 + a_2^2}{2} - c(a_1 + a_2) + c^2 \\
&= \frac{2c^2 + 16kc + 12k^2}{2} - c(2c + 4k) + c^2 \\
&= \cancel{c^2} + 8kc + 6k^2 - \cancel{2c^2} - 4kc + \cancel{c^2} \\
&= 4kc + 6k^2 \\
\implies p^2 &= 2kc + 3k^2
\end{aligned}$$

(c) We need to show that the conditions in the second part of **(i)** are met, i.e. $r^2 = k(2a-k)$ and $a > k > 0$, but in this case we have d instead of a and the radius is p.

Since we have $d = c + 2k$ and c, k are positive then we have $d > k > 0$. How to show c is positive?

Considering p^2 we have:

$$\begin{aligned}
p^2 &= 2kc + 3k^2 \\
&= k(2c + 3k) \\
&= k\big[2(d - 2k) + 3k\big] \quad \text{using } d = c + 2k \\
&= k(2d - k)
\end{aligned}$$

which is the required form, hence the circle touches the parabola tangentially at two points.

That felt like a very long question!

Question 7

7 The differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 2x\frac{\mathrm{d}x}{\mathrm{d}t}$$

describes the motion of a particle with position $x(t)$ at time t. At $t = 0$, $x = a$, where $a > 0$.

(i) Solve the differential equation in the case where $\dfrac{\mathrm{d}x}{\mathrm{d}t} = a^2$ when $t = 0$.

What happens to the particle as t increases from 0?

(ii) Solve the differential equation in the case where $\dfrac{\mathrm{d}x}{\mathrm{d}t} = a^2 + p^2$ when $t = 0$, where $p > 0$.

What happens to the particle as t increases from 0?

(iii) Solve the differential equation in the case where $\dfrac{\mathrm{d}x}{\mathrm{d}t} = a^2 - q^2$ when $t = 0$, where $q > 0$.

What happens to the particle as t increases from 0? Give conditions on a and q for the different cases which arise.

Examiner's report

While there were several very good responses to this question, there were also a significant number of candidates who did not recognise that the first differential equation could easily be turned into a first-order differential equation. Where the correct solution method was not identified, responses often did not make any further progress with the question, and many responses were very brief before the candidate opted to move on to a different question.

Those who recognised the method that was needed were often able to solve part **(i)** well although many candidates appeared to assume that the information $\dfrac{\mathrm{d}x}{\mathrm{d}t} = a^2$ at $t = 0$ would mean that $\dfrac{\mathrm{d}x}{\mathrm{d}t} = x^2$ for all t.

In part **(ii)** many candidates appeared to be familiar with the form of the required integral and so were able to reach a solution of the differential equation, although many struggled to explain the initial motion sufficiently clearly.

Part **(iii)** was generally answered well by those that attempted it, but in many cases the absolute value signs within the integrals was not dealt with sufficiently clearly within the solution. Many responses to this part of the question did not consider all of the possible cases, with the case where $a = q$ being the most commonly omitted.

Throughout the question, responses often attempted to explain the motion of the particle as $t \to \infty$, rather than the motion as t increases from 0.

Solution

(i) We have:

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= 2x\frac{\mathrm{d}x}{\mathrm{d}t}\\ &= \frac{\mathrm{d}\left(x^2\right)}{\mathrm{d}t}\\ \implies \frac{\mathrm{d}x}{\mathrm{d}t} &= x^2 + c\end{aligned}$$

When $t = 0$ we have $x = a$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = a^2$ which gives $c = 0$.

So:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= x^2\\ \int x^{-2}\,\mathrm{d}x &= \int \mathrm{d}t\\ -x^{-1} &= t + k\\ t = 0 \implies -a^{-1} &= k\\ \implies -x^{-1} &= t - a^{-1}\\ \implies -\frac{1}{x} &= \frac{at-1}{a}\\ \implies x &= \frac{a}{1-at}\end{aligned}$$

As t increases from 0, $1 - at$ decreases from 1 so x increases from a, i.e. the particle moves away from the origin.

(ii) As before we have $\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 + c$, and as $\frac{\mathrm{d}x}{\mathrm{d}t} = a^2 + p^2$ when $t = 0$ we have $c = p^2$.

Therefore:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= x^2 + p^2\\ \int \frac{1}{x^2+p^2}\,\mathrm{d}x &= t + k\\ \frac{1}{p}\arctan\left(\frac{x}{p}\right) &= t + k\\ t = 0 \implies k &= \frac{1}{p}\arctan\left(\frac{a}{p}\right)\\ \implies \frac{1}{p}\arctan\left(\frac{x}{p}\right) &= t + \frac{1}{p}\arctan\left(\frac{a}{p}\right)\\ \arctan\left(\frac{x}{p}\right) &= pt + \arctan\left(\frac{a}{p}\right)\\ x &= p\tan\left[pt + \arctan\left(\frac{a}{p}\right)\right]\end{aligned}$$

As t increases from 0, x increases, i.e. the particle moves away from the origin.

(iii) In this case we have:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x^2 - q^2 \\ \implies \int \frac{1}{x^2-q^2}\,\mathrm{d}x &= t+k \end{aligned}$$

Using partial fractions we have:

$$\begin{aligned} \frac{1}{x^2-q^2} &= \frac{1}{(x-q)(x+q)} \\ &= \frac{A}{x-q} + \frac{B}{x+q} \\ \implies 1 &= A(x+q) + B(x-q) \\ \implies A &= \frac{1}{2q} \text{ and } B = -\frac{1}{2q} \\ \implies \frac{1}{x^2-q^2} &= \frac{1}{2q(x-q)} - \frac{1}{2q(x+q)} \end{aligned}$$

Using this gives:

$$\begin{aligned} \int \frac{1}{x^2-q^2}\,\mathrm{d}x &= t+k \\ \int \frac{2q}{x^2-q^2}\,\mathrm{d}x &= 2qt+2qk \\ \int \frac{1}{(x-q)} - \frac{1}{(x+q)}\,\mathrm{d}x &= 2qt+2qk \\ \ln(x-q) - \ln(x+q) &= 2qt+2qk \\ \frac{x-q}{x+q} &= \mathrm{e}^{2qt+2qk} \\ \frac{x-q}{x+q} &= A\mathrm{e}^{2qt} \\ t=0 \implies \frac{x-q}{x+q} &= \frac{a-q}{a+q}\mathrm{e}^{2qt} \end{aligned}$$

Rearranging gives:

$$\begin{aligned} (x-q)(a+q) &= (x+q)(a-q)\mathrm{e}^{2qt} \\ x\left[(a+q)-(a-q)\mathrm{e}^{2qt}\right] &= q(a-q)\mathrm{e}^{2qt} + q(a+q) \\ x &= \frac{q(a-q)\mathrm{e}^{2qt} + q(a+q)}{(a+q)-(a-q)\mathrm{e}^{2qt}} \\ x &= \frac{q(a-q)\mathrm{e}^{2qt} - q(a+q) + 2q(a+q)}{(a+q)-(a-q)\mathrm{e}^{2qt}} \\ &= -q + \frac{2q(a+q)}{(a+q)-(a-q)\mathrm{e}^{2qt}} \end{aligned}$$

If $q=a$ then $x=a$ and the particle does not move.

If $q<a$ then $(a+q)-(a-q)\mathrm{e}^{2qt}$ is decreasing and x increases.

If $q>a$ then $(a+q)-(a-q)\mathrm{e}^{2qt}$ is increasing and x decreases.

Question 8

8 If we split a set S of integers into two subsets A and B whose intersection is empty and whose union is the whole of S, and such that

- the sum of the elements of A is equal to the sum of the elements of B
- and the sum of the squares of the elements of A is equal to the sum of the squares of the elements of B,

then we say that we have found a *balanced partition* of S into two subsets.

(i) Find a balanced partition of the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ into two subsets A and B, each of size 4.

(ii) Given that $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$ are sequences with

$$\sum_{k=1}^{m} a_k = \sum_{k=1}^{m} b_k \quad \text{and} \quad \sum_{k=1}^{m} a_k^2 = \sum_{k=1}^{m} b_k^2,$$

show that

$$\sum_{k=1}^{m} a_k^3 + \sum_{k=1}^{m} (c + b_k)^3 = \sum_{k=1}^{m} b_k^3 + \sum_{k=1}^{m} (c + a_k)^3$$

for any real number c.

(iii) Find, with justification, a balanced partition of the set $\{1, 2, 3, \ldots, 16\}$ into two subsets A and B, each of size 8, which also has the property that

- the sum of the cubes of the elements of A is equal to the sum of the cubes of the elements of B.

(iv) You are given that the sets $A = \{1, 3, 4, 5, 9, 11\}$ and $B = \{2, 6, 7, 8, 10\}$ form a balanced partition of the set $\{1, 2, 3, \ldots, 11\}$.

Let $S = \{n^2, (n+1)^2, (n+2)^2, \ldots, (n+11)^2\}$, where n is any positive integer. Find, with justification, two subsets C and D of S whose intersection is empty and whose union is the whole of S, and such that

- the sum of the elements of C is equal to the sum of the elements of D.

Examiner's report

Of the Pure questions, this attracted a relatively small number of responses, although many good solutions were seen.

Many candidates were able to write down the partition of the set as their answer to part **(i)** without much supporting working and this was awarded full marks. Part **(ii)** was answered well by most candidates, but many responses did not give sufficiently clear explanations. In particular, some simply produced the two binomial expansions and then claimed that the result would be true.

A small number of candidates attempted to solve part **(iii)** without using the result from part **(ii)**. Such attempts were rarely successful. Of those who applied the result from part **(ii)**, many did not show that the properties of a balanced partition would be satisfied by their solution. Some candidates simply showed that these were true by calculating the values for the specific case rather than showing a more general result.

In part **(iv)** many candidates recognised the need to include 0 in the set and then deduced a correct partition. However, in many cases there was insufficient justification that the two sets would have the required property.

Solution

(i) The total of all the numbers is 36, so the sum of the elements in each set must be 18. The sum of the squares is 204, so the sum of the squares of the elements in each set must be 102.

The largest square is 64 and the next largest are 49 and 36. We have $64 + 49 > 102$, and $64 + 36 = 100$ which leave a gap of 2 which cannot be made from the square numbers. Therefore we must have 8 on one set (A) and 6 and 7 in the other (B).

This leaves 17 for the sum of the squares in set B, which can only be made by taking $1 + 16$. Therefore the balanced partition is:

$$A = \{2, 3, 5, 8\}$$
$$B = \{1, 4, 6, 7\}$$

It is a good idea to just do a quick numerical check to see if these are indeed balanced!

(ii) We have:

$$\begin{aligned}
&\sum_{k=1}^{m} a_k^3 + \sum_{k=1}^{m} (c + b_k)^3 \\
&= \sum_{k=1}^{m} a_k^3 + \sum_{k=1}^{m} \left(c^3 + 3c^2 b_k + 3c b_k^2 + b_k^3\right) \\
&= \sum_{k=1}^{m} a_k^3 + \sum_{k=1}^{m} c^3 + 3c^2 \sum_{k=1}^{m} b_k + 3c \sum_{k=1}^{m} b_k^2 + \sum_{k=1}^{m} b_k^3 \\
&= \sum_{k=1}^{m} a_k^3 + \sum_{k=1}^{m} c^3 + 3c^2 \sum_{k=1}^{m} a_k + 3c \sum_{k=1}^{m} a_k^2 + \sum_{k=1}^{m} b_k^3 \\
&= \sum_{k=1}^{m} (c + a_k)^3 + \sum_{k=1}^{m} b_k^3 \qquad \text{as required.}
\end{aligned}$$

(iii) It feels as if the previous part(s) should be helpful here, and part **(ii)** brings in the idea of adding a constant to the terms.

We already know that we can find a balanced partition of $\{1,2,3,4,5,6,7,8\}$. Take:

$$a_k = \{2,3,5,8\}$$
$$b_k = \{1,4,6,7\}$$

Consider $\{9,10,11,12,13,14,15,16\} = \{1+8, 2+8, 3+8, 4+8, 5+8.6+8, 7+8, 8+8\}$.

If we take a_k and b_k as above then we have:

$$\sum_{k=1}^{4} a_k = \sum_{k=1}^{4} b_k \quad \text{and} \quad \sum_{k=1}^{4} a_k^2 = \sum_{k=1}^{4} b_k^2$$

Consider:

$$\sum_{k=1}^{4}(a_k+8) = \sum_{k=1}^{4} a_k + \sum_{k=1}^{4} 8 = \sum_{k=1}^{4}(b_k+8)$$

and:

$$\begin{aligned}\sum_{k=1}^{4}(a_k+8)^2 &= \sum_{k=1}^{4}\left(a_k^2+16a_k+64\right)\\ &= \sum_{k=1}^{4} a_k^2 + 16\sum_{k=1}^{4} a_k + \sum_{k=1}^{4} 64\\ &= \sum_{k=1}^{4}\left(b_k^2+16b_k+64\right)\\ &= \sum_{k=1}^{4}(b_k+8)^2\end{aligned}$$

Let:

$$A = \{2,3,5,8\}$$
$$B = \{1,4,6,7\}$$
$$C = \{10,11,13,16\}$$
$$D = \{9,12,14,15\}$$

Then we know that A and B are a balanced partition of $\{1,2,3,4,5,6,7,8\}$ and that C and D are a balanced partition of $\{9,10,11,12,13,14,15,16\}$.

By part **(ii)** we know that the sum of the cubes of the elements of A plus the sum of the cubes of the elements of D is equal to the sum of the cubes of the elements of the other two sets.

Therefore a balanced partition, with the additional property that the sum of the cubes are the same, is:

$$A' = \{2,3,5,8,9,12,14,15\}$$
$$B' = \{1,4,6,7,10,11,13,16\}$$

(iv) First note that A and B have different numbers of elements in them, and combined they have 11 elements where as set S has 12 elements.

Adding in element 0 we can say that the sets $A = \{1, 3, 4, 5, 9, 11\}$ and $B' = \{0, 2, 6, 7, 8, 10\}$ form a balanced partition of the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

From the previous part we know that $\{n+1, n+3, n+4, n+5, n+9, n+11\}$ and $\{n, n+2, n+6, n+7, n+8, n+10\}$ will also form a balanced partition of the set $\{n, n+1, \cdots, n+11\}$.

Therefore we can take:

$$C = \{(n+1)^2, (n+3)^2, (n+4)^2, (n+5)^2, (n+9)^2, (n+11)^2\}$$
$$\text{and } D = \{n^2, (n+2)^2, (n+6)^2, (n+7)^2, (n+8)^2, (n+10)^2\}$$

which satisfies the condition that the sum of the elements of C is equal to the sum of the elements of D.

Question 9

9 Points A and B are at the same height and a distance $\sqrt{2}\,r$ apart. Two small, spherical particles of equal mass, P and Q, are suspended from A and B, respectively, by light inextensible strings of length r. Each particle individually may move freely around and inside a circle centred at the point of suspension.

The particles are projected simultaneously from points which are a distance r vertically below their points of suspension, directly towards each other and each with speed u. When the particles collide, the coefficient of restitution in the collision is e.

(i) Show that, immediately after the collision, the horizontal component of each particle's velocity has magnitude $\frac{1}{2}ev\sqrt{2}$, where $v^2 = u^2 - gr\left(2-\sqrt{2}\right)$ and write down the vertical component in terms of v.

(ii) Show that the strings will become taut again at a time t after the collision, where t is a non-zero root of the equation

$$(r - evt)^2 + \left(-r + vt - \tfrac{1}{2}\sqrt{2}gt^2\right)^2 = 2r^2.$$

(iii) Show that, in terms of the dimensionless variables

$$z = \frac{vt}{r} \quad \text{and} \quad c = \frac{\sqrt{2}v^2}{rg}$$

this equation becomes

$$\left(\frac{z}{c}\right)^3 - 2\left(\frac{z}{c}\right)^2 + \left(\frac{2}{c} + 1 + e^2\right)\left(\frac{z}{c}\right) - \frac{2}{c}(1+e) = 0.$$

(iv) Show that, if this equation has three equal non-zero roots, $e = \frac{1}{3}$ and $v^2 = \frac{9}{2}\sqrt{2}rg$. Explain briefly why, in this case, no energy is lost when the string becomes taut.

(v) In the case described in **(iv)**, the particles have speed U when they again reach the points of their motion vertically below their points of suspension. Find U^2 in terms of r and g.

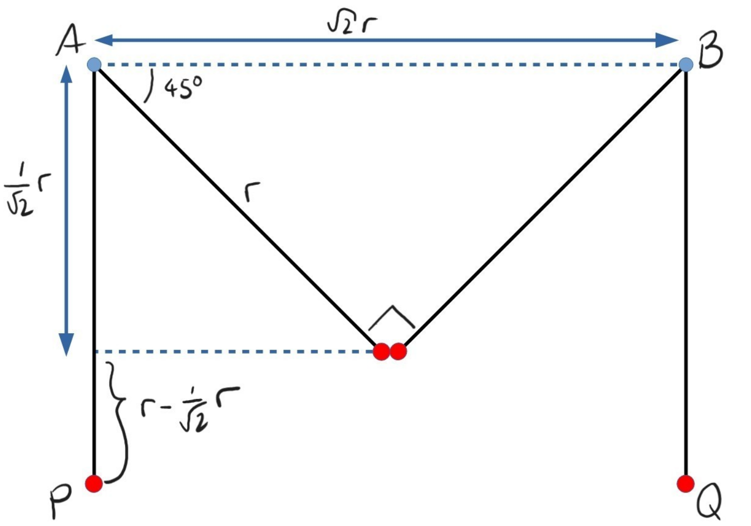
Examiner's report

Only a small number of candidates attempted this question and many were not able to set up the problem sufficiently well to make good progress. Those who recognised that conservation of energy could be used in part **(i)** were often able to reach the given results successfully, although many assumed that the vertical component of the velocity would also change during the collision.

Both parts **(ii)** and **(iii)** were well answered by those that attempted them, with errors in the algebra being the main cause of marks being lost. Very few of those who attempted part **(iv)** were able to explain why no energy is lost when the string becomes taut again, but most were then able to produce good solutions to part **(v)**.

Solution

Diagram time!



Note that the situation is symmetrical, and so when the particles collide it will be on the line which is the perpendicular bisector of the line segment AB. Since the situation is symmetrical we can probably just consider what happens to particle P, as Q will just mirror what P does.

(i) Let the mass of each particle be m, and let the speed of each particle just before they collide be equal to v. Taking the height at which the particles are at rest before the motion starts as where gravitational potential energy is equal to 0, then conservation of energy gives:

$$2 \times \frac{1}{2}mu^2 = 2 \times mg\left(1 - \frac{1}{\sqrt{2}}\right)r + 2 \times \frac{1}{2}mv^2$$
$$u^2 = 2g\left(1 - \frac{1}{\sqrt{2}}\right)r + v^2$$
$$\implies v^2 = u^2 - gr(2 - \sqrt{2})$$

The velocity of particle P just before the collision is given by $\begin{pmatrix} \frac{v}{\sqrt{2}} \\ \frac{v}{\sqrt{2}} \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} v \\ v \end{pmatrix}$ (and similarly for Q but with a negative horizontal velocity). After the collision has happened the vertical components of the velocities will be unchanged i.e. equal to $\frac{\sqrt{2}}{2}v$.

The horizontal speed of approach is given by $\sqrt{2}v$, and so by Newton's law of restitution the speed of separation is equal to $\sqrt{2}ev$. By symmetry the horizontal velocities are equal is magnitude but opposite in direction, so the horizontal components of the velocities of P and Q will be $-\frac{1}{2}ev\sqrt{2}$ and $\frac{1}{2}ev\sqrt{2}$ respectively.

Initially I just wrote down the horizontal components of the speed, but since the question is a "show that" slightly more justification is needed.

(ii) After the collision P's motion relative to the point A is given by:

$$\begin{pmatrix} \frac{1}{2}r\sqrt{2} - \frac{1}{2}ev\sqrt{2}t \\ -\frac{1}{2}r\sqrt{2} + \frac{1}{2}vt\sqrt{2} - \frac{1}{2}gt^2 \end{pmatrix}$$

The string will become taut again when the distance from A is equal to r, i.e. when:

$$\begin{aligned} \left(\tfrac{1}{2}r\sqrt{2} - \tfrac{1}{2}ev\sqrt{2}t\right)^2 + \left(-\tfrac{1}{2}r\sqrt{2} + \tfrac{1}{2}vt\sqrt{2} - \tfrac{1}{2}gt^2\right)^2 &= r^2 \\ \tfrac{1}{2}\left(r - evt\right)^2 + \tfrac{1}{2}\left(-r + vt - \tfrac{\sqrt{2}}{2}gt^2\right)^2 &= r^2 \\ \implies \left(r - evt\right)^2 + \left(-r + vt - \tfrac{\sqrt{2}}{2}gt^2\right)^2 &= 2r^2 \end{aligned}$$

(iii) Dividing throughout by r^2 gives:

$$\begin{aligned} \left(r - evt\right)^2 + \left(-r + vt - \tfrac{\sqrt{2}}{2}gt^2\right)^2 &= 2r^2 \\ \left(1 - e\frac{vt}{r}\right)^2 + \left(-1 + \frac{vt}{r} - \frac{\sqrt{2}gt^2}{2r}\right)^2 &= 2 \\ (1 - ez)^2 + \left(-1 + z - \frac{\sqrt{2}g}{2r} \times \left(\frac{zr}{v}\right)^2\right)^2 &= 2 \\ (1 - ez)^2 + \left(-1 + z - \frac{gz^2r}{\sqrt{2}v^2}\right)^2 &= 2 \\ (1 - ez)^2 + \left(-1 + z - \frac{z^2}{c}\right)^2 &= 2 \\ \cancel{1} - 2ez + e^2z^2 + \cancel{1} + z^2 + \frac{z^4}{c^2} - 2z - 2\frac{z^3}{c} + 2\frac{z^2}{c} &= \cancel{2} \\ z\left[\frac{z^3}{c^2} - \frac{2z^2}{c} + \left(\frac{2z}{c} + z + e^2z\right) - 2e - 2\right] &= 0 \end{aligned}$$

$z = 0$ corresponds to $t = 0$, i.e. when the particles collide. So when the string becomes taut again we have:

$$\begin{aligned} \frac{z^3}{c^2} - \frac{2z^2}{c} + \left(\frac{2z}{c} + z + e^2z\right) - 2e - 2 &= 0 \\ \implies \left(\frac{z}{c}\right)^3 - 2\left(\frac{z}{c}\right)^2 + \left(\frac{2}{c} + 1 + e^2\right)\left(\frac{z}{c}\right) - \frac{2}{c}(e+1) &= 0 \end{aligned}$$

(iv) If there are three equal roots for $\frac{z}{c}$, then the sum of these is equal to 2, so $\alpha = \frac{2}{3}$. Considering the coefficient of $\frac{z}{c}$ and the constant Vieta's formulae[2] give:

$$\frac{2}{c} + 1 + e^2 = 3\left(\frac{2}{3}\right)^2$$

$$\text{and} \quad \frac{2}{c}(e+1) = \left(\frac{2}{3}\right)^3$$

$$\implies e + 1 = \frac{c}{2} \times \frac{8}{27}$$

$$\implies c = \frac{27}{4}(e+1)$$

$$\text{Substituting gives} \quad \frac{8}{27(e+1)} + 1 + e^2 = 3\left(\frac{2}{3}\right)^2$$

$$8 + 27(1+e^2)(e+1) = 36(e+1)$$

$$8 + 27 + 27e + 27e^2 + 27e^3 = 36e + 36$$

$$27e^3 + 27e^2 - 9e - 1 = 0$$

$$(3e-1)(9e^2 + 12e + 1) = 0$$

So $e = \frac{1}{3}$, as the roots of $9e^2 + 12e + 1 = 0$ must be negative.

This gives:

$$c = \frac{27}{4}(e+1)$$

$$= \frac{27}{4}\left(\frac{1}{3} + 1\right)$$

$$= \frac{27}{4} \times \frac{4}{3} = 9$$

Therefore we have:

$$\frac{\sqrt{2}v^2}{rg} = 9$$

$$\implies v^2 = \tfrac{9}{2}\sqrt{2}rg$$

Since the root is repeated the string becomes taut at a point where the direction of motion is tangential to the circle, which means there is no jerk in the string, so no energy lost.

The question said "explain briefly" which implies that there are not many marks available for this part!

[2]Vieta's formulae are the equations relating the coefficients of a polynomial to the sums and products of the roots. It is quicker to write "Vieta implies" than "using the sum and products of roots we have".

(v) Using conservation of energy for particle P (or Q) we have:

$$\begin{aligned}
\tfrac{1}{2}m\left[\left(\tfrac{1}{2}ev\sqrt{2}\right)^2+\left(\tfrac{1}{2}v\sqrt{2}\right)^2\right]+mgr\left(1-\tfrac{1}{\sqrt{2}}\right)&=\tfrac{1}{2}mU^2\\
\left(\tfrac{1}{6}v\sqrt{2}\right)^2+\left(\tfrac{1}{2}v\sqrt{2}\right)^2+gr\left(2-\sqrt{2}\right)&=U^2\\
\frac{1}{18}v^2+\frac{1}{2}v^2+gr\left(2-\sqrt{2}\right)&=U^2\\
\frac{10}{18}v^2+gr\left(2-\sqrt{2}\right)&=U^2\\
\frac{5}{9}\times\frac{9}{2}\sqrt{2}rg+gr\left(2-\sqrt{2}\right)&=U^2\\
\left(2+\frac{3\sqrt{2}}{2}\right)rg&=U^2
\end{aligned}$$

Question 10

10 The lower end of a rigid uniform rod of mass m and length a rests at point M on rough horizontal ground. Each of two elastic strings, of natural length ℓ and modulus of elasticity λ, is attached at one end to the top of the rod. Their lower ends are attached to points A and B on the ground, which are a distance $2a$ apart. M is the midpoint of AB.

P is the point at the top of the rod and lies in the vertical plane through AMB.

Suppose that the rod is in equilibrium with angle $PMB = 2\theta$, where $\theta < 45°$ and ℓ is such that both strings are in tension.

(i) Show that angle APB is a right angle.

Show that that the force exerted on the rod by the elastic strings can be written as the sum of

- a force of magnitude $\dfrac{2a\lambda}{\ell}$ parallel to the rod
- and a force of magnitude $\sqrt{2}\,\lambda$ acting along the bisector of angle APB.

(ii) By taking moments about point M, or otherwise, show that $\cos\theta + \sin\theta = \dfrac{2\lambda}{mg}$.

Deduce that it is necessary that $\frac{1}{2}mg < \lambda < \frac{1}{2}\sqrt{2}\,mg$.

(iii) N and F are the magnitudes of the normal and frictional forces, respectively, exerted on the rod by the ground at M.

Show, by taking moments about an appropriate point, or otherwise, that

$$N - F\tan 2\theta = \tfrac{1}{2}mg.$$

Examiner's report

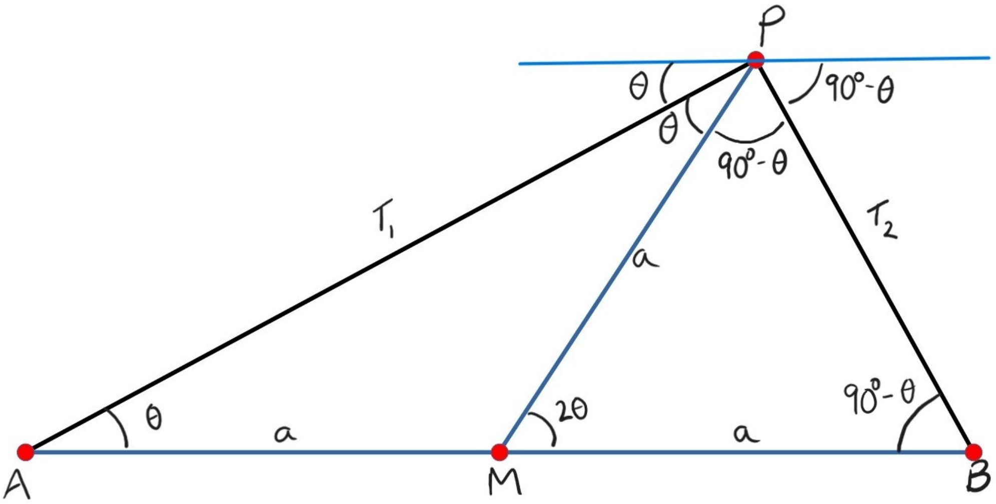
As with Question **9**, very few candidates attempted this question, and those that did were often unable to set up the problem sufficiently well to make much progress.

A significant number of candidates were able to produce a geometric argument to show that the angle is a right angle in part **(i)**, but many then did not produce correct expressions for the tension in the two strings. Many candidates struggled with the concept of resolving forces in two non-orthogonal directions and so struggled to make any progress beyond this point.

The small number of candidates who were able to complete part **(i)** often managed to solve the remaining two parts of the question well.

Solution

First thing to do is to draw a diagram:



(i) We know that triangle AMP is isosceles and hence angles MAP and MPA are both equal, and since they must sum to the exterior angle BMP they must both be equal to to θ.

Since triangle MBP is also isosceles angle we have $MPB = \frac{1}{2}(180° - 2\theta) = 90° - \theta$. Hence $\angle APB = \angle APM + \angle MPB = \theta + 90° - \theta = 90°$.

Using trigonometry in triangle APB we have $PA = 2a\cos\theta$ and $PB = 2a\sin\theta$. Using Hooke's law we have:

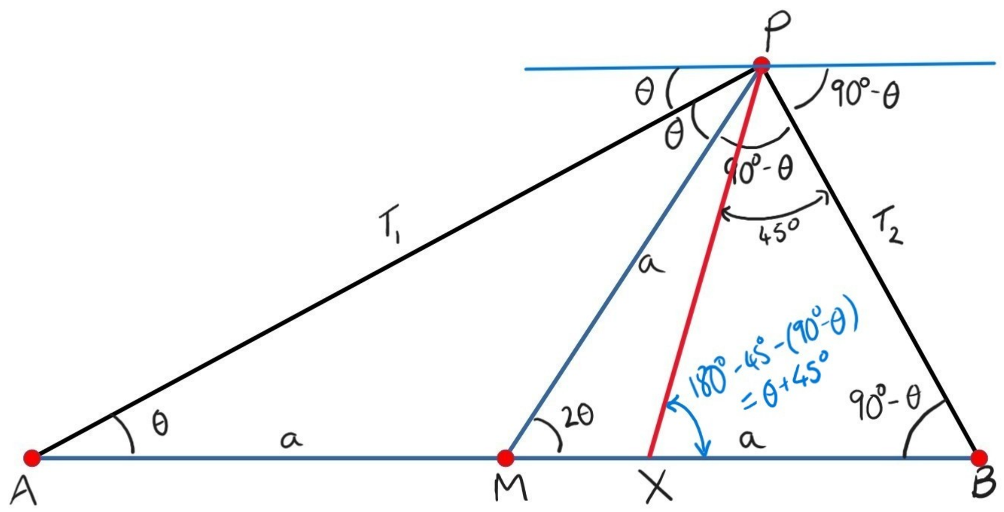
$$T_1 = \frac{\lambda(2a\cos\theta - l)}{l}$$
$$T_2 = \frac{\lambda(2a\sin\theta - l)}{l}$$

Resolving the tensions horizontally and vertically the sum is given as:

$$\begin{aligned}
&\frac{\lambda(2a\cos\theta - l)}{l}\begin{pmatrix}-\cos\theta \\ -\sin\theta\end{pmatrix} + \frac{\lambda(2a\sin\theta - l)}{l}\begin{pmatrix}\cos(90° - \theta) \\ -\sin(90° - \theta)\end{pmatrix} \\
=&\frac{\lambda(2a\cos\theta - l)}{l}\begin{pmatrix}-\cos\theta \\ -\sin\theta\end{pmatrix} + \frac{\lambda(2a\sin\theta - l)}{l}\begin{pmatrix}\sin\theta \\ -\cos\theta\end{pmatrix} \\
=&\frac{2a\lambda}{l}\begin{pmatrix}\sin^2\theta - \cos^2\theta \\ -2\sin\theta\cos\theta\end{pmatrix} + \lambda\begin{pmatrix}\cos\theta - \sin\theta \\ \sin\theta + \cos\theta\end{pmatrix} \\
=&\frac{2a\lambda}{l}\begin{pmatrix}-\cos 2\theta \\ -\sin 2\theta\end{pmatrix} + \sqrt{2}\lambda\begin{pmatrix}\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta \\ \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta\end{pmatrix} \\
=&\frac{2a\lambda}{l}\begin{pmatrix}-\cos 2\theta \\ -\sin 2\theta\end{pmatrix} + \sqrt{2}\lambda\begin{pmatrix}\cos\left(\theta + \frac{\pi}{4}\right) \\ \sin\left(\theta + \frac{\pi}{4}\right)\end{pmatrix}
\end{aligned}$$

We can see that the force with magnitude $\dfrac{2a\lambda}{l}$ acts in the direction of the rod (as the angle this force makes with the horizontal is equal to 2θ).

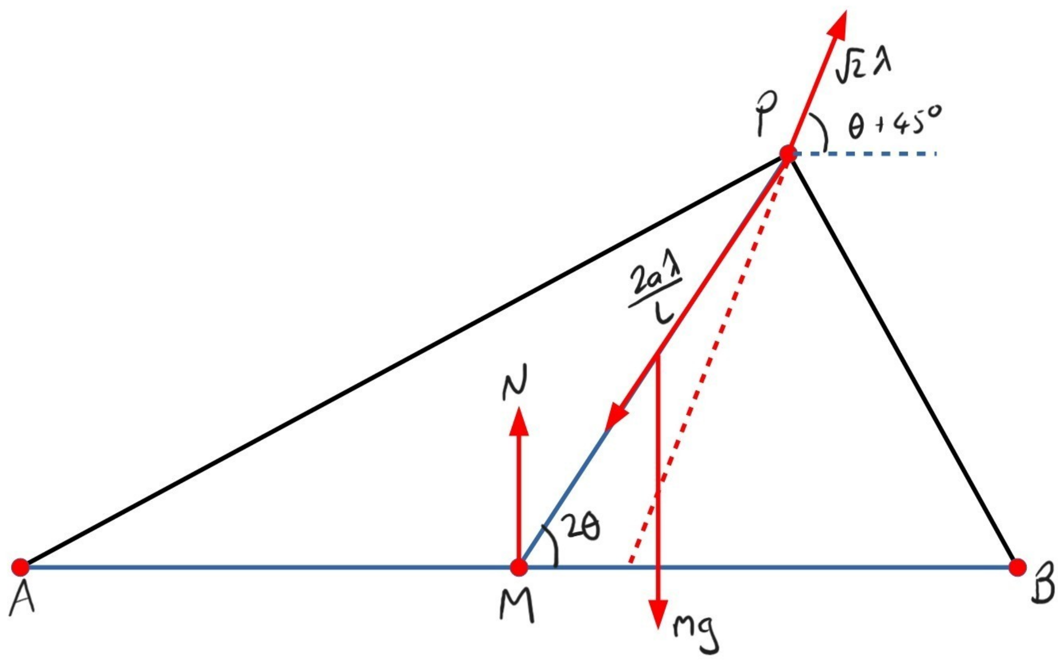
Since the angle APB is $90°$ then the angle bisector makes an angle of $45°$ with the elastic strings.



From the diagram above we can see that the angle bisector makes an angle of $\theta + 45°$ with the horizontal, so the force with magnitude $\sqrt{2}\lambda$ acts in this direction.

(ii) The first request seemed a little unusual, but it does make taking moments about M a little easier! Remember that you can still attempt part **(ii) even if you didn't manage to finish part (i).**

The forces acting on the rod are shown below:



Using moments about M we have:

$$mg \times \tfrac{1}{2}a\cos 2\theta = \sqrt{2}\lambda \times a\cos(\theta + 45^\circ)$$
$$\tfrac{1}{2}mg\left(\cos^2\theta - \sin^2\theta\right) = \lambda(\cos\theta - \sin\theta)$$
$$\cos^2\theta - \sin^2\theta = \frac{2\lambda}{mg}(\cos\theta - \sin\theta)$$

Since $\theta < 45^\circ$ we have $\cos\theta > \sin\theta$ and so we can divide by $\cos\theta - \sin\theta$ to get:

$$\cos\theta + \sin\theta = \frac{2\lambda}{mg}$$

as required.

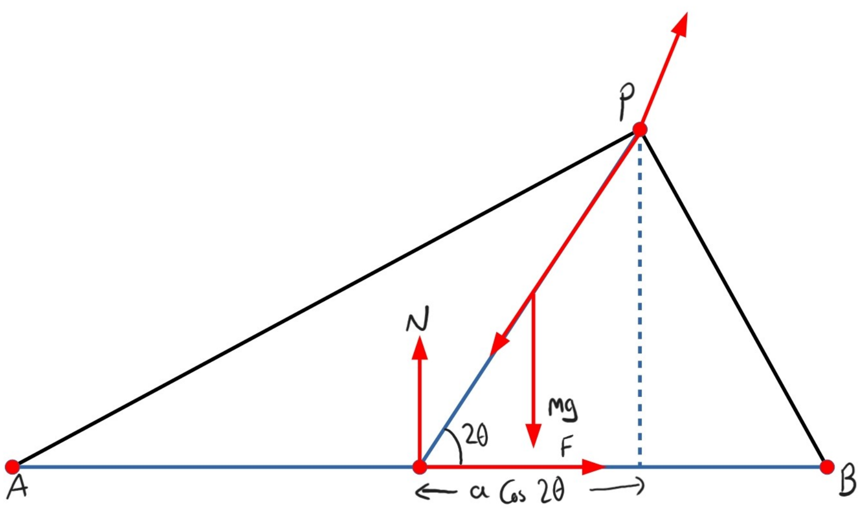
Using $\cos\theta + \sin\theta = R\sin(\theta + \alpha)$ we have:

$$\cos\theta + \sin\theta = \sqrt{2}\sin(\theta + 45^\circ)$$

Since $0 < \theta < 45^\circ$ we have:

$$\sqrt{2}\sin(45^\circ) < \frac{2\lambda}{mg} < \sqrt{2}\sin(45^\circ + 45^\circ)$$
$$1 < \frac{2\lambda}{mg} < \sqrt{2}$$
$$\tfrac{1}{2}mg < \lambda < \tfrac{1}{2}\sqrt{2}mg$$

(iii) Taking moments about P we have:



$$\tfrac{1}{2}amg\cos 2\theta + Fa\sin 2\theta = Na\cos 2\theta$$
$$\tfrac{1}{2}mg = N - F\tan 2\theta$$

Alternative method:

Equating vertical forces gives:

$$N + \sqrt{2}\lambda \sin(\theta + 45^\circ) = mg + \frac{2a\lambda}{l}\sin 2\theta$$

and equating horizontal forces:

$$\frac{2a\lambda}{l}\cos 2\theta = F + \sqrt{2}\lambda\cos(\theta + 45^\circ)$$

Substituting for $\dfrac{2a\lambda}{l}$ gives:

$$N + \sqrt{2}\lambda \sin(\theta + 45^\circ) = mg + \left(\frac{F + \sqrt{2}\lambda\cos(\theta + 45^\circ)}{\cos 2\theta}\right)\sin 2\theta$$

$$N - F\tan 2\theta = mg + \frac{\sqrt{2}\lambda\cos(\theta + 45^\circ)}{\cos 2\theta}\sin 2\theta - \sqrt{2}\lambda\sin(\theta + 45^\circ)$$

$$N - F\tan 2\theta = mg + \frac{\sqrt{2}\lambda}{\cos 2\theta}\Big[\cos(\theta + 45^\circ)\sin 2\theta - \sin(\theta + 45^\circ)\cos 2\theta\Big]$$

$$N - F\tan 2\theta = mg + \frac{\sqrt{2}\lambda}{\cos 2\theta}\left[\frac{1}{\sqrt{2}}(\cos\theta - \sin\theta)\sin 2\theta - \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta)\cos 2\theta\right]$$

$$N - F\tan 2\theta = mg + \frac{mg(\sin\theta + \cos\theta)}{2\cos 2\theta}\Big[(\cos\theta - \sin\theta)\sin 2\theta - (\cos\theta + \sin\theta)\cos 2\theta\Big]$$

$$N - F\tan 2\theta = mg + \frac{mg(s + c)}{2(c^2 - s^2)}\Big[2c^2 s - 2s^2 c - c^3 - sc^2 + cs^2 + s^3\Big]$$

$$N - F\tan 2\theta = mg + \frac{mg(s + c)}{2(c^2 - s^2)}\Big[c^2 s - s^2 c - c^3 + s^3\Big]$$

$$N - F\tan 2\theta = mg + \frac{mg}{2(c - s)}\Big[s(c^2 + s^2) - c(s^2 + c^2)\Big]$$

$$N - F\tan 2\theta = mg - \frac{mg}{2}$$

$$N - F\tan 2\theta = \frac{mg}{2}$$

Note that the suggested method was a lot quicker!

Question 11

11 **(i)** By considering the sum of a geometric series, or otherwise, show that

$$\sum_{r=1}^{\infty} r x^{r-1} = \frac{1}{(1-x)^2} \quad \text{for } |x| < 1.$$

(ii) Ali plays a game with a fair $2k$-sided die. He rolls the die until the first $2k$ appears. Ali wins if all the numbers he rolls are even.

(a) Find the probability that Ali wins the game.

If Ali wins the game, he earns £1 for each roll, including the final one. If he loses, he earns nothing.

(b) Find Ali's expected earnings from playing the game.

(iii) Find a simplified expression for

$$1 + 2\binom{n}{1}x + 3\binom{n}{2}x^2 + \ldots + (n+1)x^n,$$

where n is a positive integer.

(iv) Zen plays a different game with a fair $2k$-sided die. She rolls the die until the first $2k$ appears, and wins if the numbers rolled are strictly increasing in size. For example, if $k = 3$, she wins if she rolls 2, 6 or 1, 4, 5, 6, but not if she rolls 1, 4, 2, 6 or 1, 3, 3, 6.

If Zen wins the game, she earns £1 for each roll, including the final one. If she loses, she earns nothing.

Find Zen's expected earnings from playing the game.

(v) Using the approximation

$$\left(1 + \frac{1}{n}\right)^n \approx \mathrm{e} \quad \text{for large } n,$$

show that, when k is large, Zen's expected earnings are a little over 35% more than Ali's expected earnings.

Examiner's report

A large number of very good solutions to this question were seen.

Part **(i)** was completed well by the majority of candidates. However, many candidates did not identify the correct probabilities to use in the calculations for part **(ii)**. Attempts at part **(ii)(b)** often applied a correct method for calculating the expected value, but used the incorrect value for the probability that had been used in part **(ii)(a)** and so gained the method mark for part **(ii)(b)**. Most candidates who attempted part **(iii)** were able to complete it successfully, usually by applying a similar approach to the one used in part **(i)**.

Combinatorial errors were common in part **(iv)**, with candidates often confusing $2k$ and $2k-1$ in their calculations or incorrectly accounting for the requirements of the order.

Part **(v)** was usually completed well by those candidates that had previously obtained the correct expressions for the expected values.

Solution

(i) Considering a geometric series with common ratio x, where $|x| < 1$ we have:

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$$

Differentiating both side with respect to x we have:

$$0 + 1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}$$

$$\implies \sum_{r=1}^{\infty} r x^{r-1} = \frac{1}{(1-x)^2}$$

(ii) **(a)** For Ali to win on the first roll, he must get $2k$ on the first roll, which has probability $\frac{1}{2k}$. For Ali to win on the second roll he must get an even number, but not $2k$, on the first roll and then $2k$ on the second roll. This has probability $\frac{k-1}{2k} \times \frac{1}{2k}$. Continuing this we have:

$$\begin{aligned}
\mathrm{P}(\text{Ali wins}) &= \frac{1}{2k} + \frac{k-1}{2k} \times \frac{1}{2k} + \left(\frac{k-1}{2k}\right)^2 \times \frac{1}{2k} + \cdots \\
&= \frac{1}{2k}\left[1 + \frac{k-1}{2k} + \left(\frac{k-1}{2k}\right)^2 + \cdots\right] \\
&= \frac{1}{2k} \times \frac{1}{1 - \frac{k-1}{2k}} \\
&= \frac{1}{2k - (k-1)} \\
&= \frac{1}{k+1}
\end{aligned}$$

Note that the geometric sum converges as we have $0 < \frac{k-1}{2k} < 1$.

(b) The expectation of Ali's earning is given by:

$$\begin{aligned}
&1 \times \frac{1}{2k} + 2 \times \frac{k-1}{2k} \times \frac{1}{2k} + 3 \times \left(\frac{k-1}{2k}\right)^2 \times \frac{1}{2k} + \cdots \\
=&\frac{1}{2k}\left[1 + 2\left(\frac{k-1}{2k}\right) + 3\left(\frac{k-1}{2k}\right)^2 + \cdots\right] \\
=&\frac{1}{2k} \times \frac{1}{\left(1 - \frac{k-1}{2k}\right)^2} \quad \text{using the result from } \textbf{(i)} \\
=&\frac{2k}{(2k-(k-1))^2} \\
=&\frac{2k}{(k+1)^2}
\end{aligned}$$

(iii) It is a reasonable assumption that this part might use a similar trick to part **(i)**. In this case the result looks like it might have something to do with the binomial expansion.

Considering $(1+x)^n$ we have:

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + x^n$$

This doesn't look as if it will give us the required result when differentiating, but if we multiply throughout by x it looks a bit more promising! We have:

$$\begin{aligned}
x(1+x)^n &= x + \binom{n}{1}x^2 + \binom{n}{2}x^3 + \binom{n}{3}x^4 + \cdots + x^{n+1} \\
\implies (1+x)^n + nx(1+x)^{n-1} &= 1 + 2\binom{n}{1}x + 3\binom{n}{2}x^2 + 4\binom{n}{3}x^3 + \cdots + (n+1)x^n \\
\implies (1+x+nx)(1+x)^{n-1} &= 1 + 2\binom{n}{1}x + 3\binom{n}{2}x^2 + 4\binom{n}{3}x^3 + \cdots + (n+1)x^n
\end{aligned}$$

and so the simplified expression is $\big[1+(n+1)x\big](1+x)^{n-1}$.

Using sigma notation this gives:

$$\sum_{r=1}^{n+1} r\binom{n}{r-1}x^{r-1} = \big[1+(n+1)x\big](1+x)^{n-1} \qquad (*)$$

(iv) It's pretty safe to assume that the previous part will be useful here!

First thing to note is that after a certain number of rolls Zen cannot possibly win, as the longest possible strictly increasing sequence is $1, 2, 3, 4, \cdots, 2k$. This means that if Zen wins she does in in $2k$ or fewer rolls.

If Zen wins in r rolls then there must be a strictly increasing sequence of $r-1$ values followed by a $2k$. There are $\binom{2k-1}{r-1}$ was of selecting $r-1$ values from the $2k-1$ values not including $2k$, and each of these ways corresponds to a unique strictly increasing sequence.

The total number of possible sequences (not just winning ones) formed by rolling the dice $r-1$ times is $(2k)^{r-1}$. Hence the probability that Zen has a strictly increasing sequence of length $r-1$ followed by $2k$ is:

$$\text{P(Zen wins after } r \text{ throws)} = \frac{1}{(2k)^{r-1}}\binom{2k-1}{r-1} \times \frac{1}{2k}$$

Zen's expected winnings are:

$$\begin{aligned}\sum_{r=1}^{2k} \frac{r}{(2k)^r}\binom{2k-1}{r-1} &= \frac{1}{2k}\sum_{r=1}^{2k} \frac{r}{(2k)^{r-1}}\binom{2k-1}{r-1}\\ &= \frac{1}{2k}\left(1+2k\times\frac{1}{2k}\right)\left(1+\frac{1}{2k}\right)^{2k-2}\\ &\qquad \left[\text{using } (*) \text{ with } n=2k-1, x=\frac{1}{2k}\right]\\ &= \frac{1}{k}\left(1+\frac{1}{2k}\right)^{2k-2}\end{aligned}$$

(v) We can rewrite Zen's winnings as:

$$\begin{aligned}\frac{1}{k}\left(1+\frac{1}{2k}\right)^{2k-2} &= \frac{1}{k}\left(1+\frac{1}{2k}\right)^{2k}\left(1+\frac{1}{2k}\right)^{-2}\\ &\approx \frac{\mathrm{e}}{k}\left(\frac{2k+1}{2k}\right)^{-2}\\ &= \frac{\mathrm{e}}{k}\left(\frac{2k}{2k+1}\right)^{2}\end{aligned}$$

Considering the ratio $\dfrac{\mathrm{E}(Z)}{\mathrm{E}(A)}$ we have:

$$\begin{aligned}\frac{\mathrm{E}(Z)}{\mathrm{E}(A)} &\approx \frac{\mathrm{e}}{k}\left(\frac{2k}{2k+1}\right)^{2}\times\frac{(k+1)^2}{2k}\\ &= 2\mathrm{e}\times\left(\frac{k+1}{2k+1}\right)^2\\ &\approx 2\mathrm{e}\times\frac{1}{4} = \frac{\mathrm{e}}{2}\end{aligned}$$

Since $\mathrm{e}\approx 2.7...$ we know that $\dfrac{\mathrm{e}}{2}$ is a little more than 1.35, and so we have:

$$\mathrm{E}(Z) \approx 1.35\mathrm{E}(A)$$

and so Zen's expected earnings are a little over 35% more than Ali's.

Question 12

12 Let X be a Poisson random variable with mean λ and let $p_r = \mathrm{P}(X = r)$, for $r = 0, 1, 2, \dots$.

Neither λ nor $\lambda + \frac{1}{2} + \sqrt{\lambda + \frac{1}{4}}$ is an integer.

(i) Show, by considering the sequence $d_r \equiv p_r - p_{r-1}$ for $r = 1, 2, \dots$, that there is a unique integer m such that $\mathrm{P}(X = r) \leqslant \mathrm{P}(X = m)$ for all $r = 0, 1, 2, \dots$, and that $\lambda - 1 < m < \lambda$.

(ii) Show that the minimum value of d_r occurs at $r = k$, where k is such that

$$k < \lambda + \tfrac{1}{2} + \sqrt{\lambda + \tfrac{1}{4}} < k + 1.$$

(iii) Show that the condition for the maximum value of d_r to occur at $r = 1$ is

$$1 < \lambda < 2 + \sqrt{2}.$$

(iv) In the case $\lambda = 3.36$, sketch a graph of p_r against r for $r = 0, 1, 2, \dots, 6, 7$.

Examiner's report

This was the least popular question on the paper. Most candidates were able to make good progress with parts **(i)** and **(ii)**, but many then struggled with the remaining parts.

In part **(i)** many candidates did not justify their handling of the inequalities or to deal properly with the fact that λ was not an integer.

Part **(ii)** similarly involved a number of attempts that did not justify the handling of the inequalities sufficiently well. Additionally, some showed that a minimum would satisfy the given conditions if it exists, but did not show that there must be a minimum.

Most of the candidates who reached part **(iii)** were able to derive the bound $\lambda < 2+\sqrt{2}$, but almost none were able to prove that $\lambda > 1$.

A number of good sketches of the graph were produced for part **(iv)**, but a significant number sketched it as a continuous curve.

Solution

The statement that λ is not an integer seems fairly normal, but the other restriction given in the stem is a little more unusual. Hopefully the reason for this restriction should become clearer when working through the question, and it might be useful as a check that we are doing the right sort of thing.

(i) We have:

$$d_r = \frac{\lambda^r e^\lambda}{r!} - \frac{\lambda^{r-1} e^\lambda}{(r-1)!}$$
$$= \frac{\lambda^{r-1} e^\lambda}{r!}(\lambda - r)$$

We know that $\frac{\lambda^{r-1} e^\lambda}{r!} > 0$ and so as long as $r < \lambda$ we will have $d_r > 0$ and when $r > \lambda$ we have $d_r < 0$. Since λ is not an integer we cannot have $r = \lambda$. This means that when $r < \lambda$ we have $p_r > p_{r-1}$ (the probabilities are increasing), and when $r > \lambda$ we have $p_r < p_{r-1}$ and the probabilities are decreasing.

Hence there is a unique maximum value of p_r which occurs when $r < \lambda$ but $r + 1 > \lambda$ i.e. $\lambda - 1 < r < \lambda$.

Note that considering the ratios $\frac{p_r}{p_{r-1}}$ is also a method that can find the maximum probability, but in this question we are told "Show, by considering the sequence..." and there is no "or otherwise" in the statement, so for this part we must use the difference between successive probabilities rather than the ratios. There is no such restriction in the next part so both differences and ratios can be used.

(ii) From the previous part, we know that d_r is positive for $r \leqslant m$ and d_r is negative for $r > m$ where m is the value found in part **(i)**, i.e. $\lambda - 1 < m < \lambda$. Therefore the minimum value of d_r must occur when $r > \lambda$.

Let $\Delta_r = d_r - d_{r-1}$:

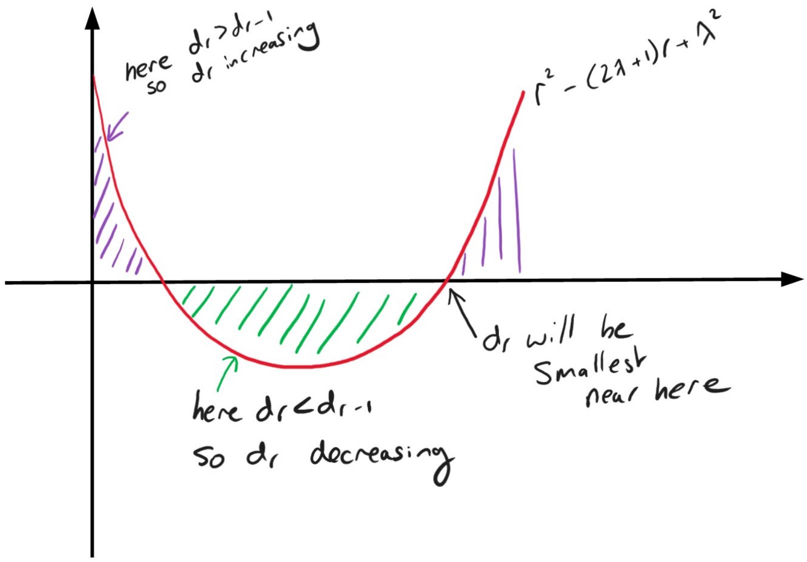
$$\Delta_r = d_r - d_{r-1}$$
$$= \frac{\lambda^{r-1} e^\lambda}{r!}(\lambda - r) - \frac{\lambda^{r-2} e^\lambda}{(r-1)!}(\lambda - (r-1))$$
$$= \frac{\lambda^{r-2} e^\lambda}{(r-1)!}\left[\frac{\lambda}{r}(\lambda - r) - (\lambda - r + 1)\right]$$
$$= \frac{\lambda^{r-2} e^\lambda}{r!}\left[\lambda^2 - \lambda r - r\lambda + r^2 - r\right]$$
$$= \frac{\lambda^{r-2} e^\lambda}{r!}\left[r^2 - (2\lambda + 1)r + \lambda^2\right]$$

Therefore whenever we have $r^2 - (2\lambda + 1)r + \lambda^2 > 0$ then $d_r > d_{r-1}$ and whenever $r^2 - (2\lambda + 1)r + \lambda^2 < 0$ then $d_r < d_{r-1}$.

Solving $r^2 - (2\lambda + 1)r + \lambda^2 = 0$ gives the values:

$$\frac{(2\lambda + 1) \pm \sqrt{(2\lambda + 1)^2 - 4\lambda^2}}{2} = \frac{(2\lambda + 1) \pm \sqrt{4\lambda + 1}}{2}$$
$$= \left(\lambda + \frac{1}{2}\right) \pm \sqrt{\lambda + \frac{1}{4}}$$

Sketching $r^2 - (2\lambda + 1)r + \lambda^2$ gives:



And so when $\left(\lambda + \frac{1}{2}\right) - \sqrt{\lambda + \frac{1}{4}} < r < \left(\lambda + \frac{1}{2}\right) + \sqrt{\lambda + \frac{1}{4}}$ we have $d_r - d_{r-1} < 0 \implies d_r < d_{r-1}$.

Hence the minimum value of d_r occurs when r is the largest possible integer in the given range, i.e. when we have:

$$k < \left(\lambda + \frac{1}{2}\right) + \sqrt{\lambda + \frac{1}{4}} < k + 1$$

Note that since $\left(\lambda + \frac{1}{2}\right) + \sqrt{\lambda + \frac{1}{4}}$ is not an integer we must have k strictly less than this.

(iii) As $r \to \infty$ the gaps between p_{r-1} and p_r tend to 0, i.e. $d_r \to 0$. Hence if d_1 is going to be the greatest difference then we need $d_1 \geqslant 0$ (as if d_1 is negative then it will be smaller than the value of d_r for a value of r close enough to infinity) . This gives:

$$\begin{aligned} d_1 &\geqslant 0 \\ \frac{\lambda^0 \mathrm{e}^\lambda}{1!}(\lambda - 1) &\geqslant 0 \\ \implies \lambda &\geqslant 1 \end{aligned}$$

We also need $d_1 > d_2$, so:

$$\begin{aligned} \frac{\lambda^0 \mathrm{e}^\lambda}{1!}(\lambda - 1) &> \frac{\lambda^1 \mathrm{e}^\lambda}{2!}(\lambda - 2) \\ \lambda - 1 &> \frac{1}{2}\lambda(\lambda - 2) \\ 2\lambda - 2 &> \lambda^2 - 2\lambda \\ \lambda^2 - 4\lambda + 2 &< 0 \end{aligned}$$

which gives $2 - \sqrt{2} < \lambda < 2 + \sqrt{2}$. Combining this with $\lambda \geqslant 1$ gives $1 \leqslant \lambda < 2 + \sqrt{2}$, **but** λ is not an integer so we have $1 < \lambda < 2 + \sqrt{2}$ as required.

(iv) If $\lambda = 3.36$ then the inequality $1 < \lambda < 2 + \sqrt{2} \approx 3.4$ is satisfied.

From part **(ii)** the minimum value of d_r occurs when $k < 3.86 + \sqrt{3.61} < k + 1$, so occurs when $r = 5$.

From part **(i)** the maximum values of p_r occurs when $2.36 < m < 3.36$, so when $r = 3$.

To sketch the graph note that:

- This has to be a discontinuous graph!
- The maximum probability occurs when $r = 3$
- The biggest jump up between two values occurs between p_0 and p_1
- The biggest jump down between two values occurs between p_4 and p_5

We now have enough to sketch a graph: